MATRIX REPRESENTATIONS OF OCTONIONS AND THEIR APPLICATIONS

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(Received: June 13, 2000; Accepted: July 20, 2000)

Abstract. As is well-known, the real quaternion division algebra $\mathbb{H}$ is algebraically isomorphic to a 4-by-4 real matrix algebra. But the real division octonion algebra $\mathbb{O}$ can not be algebraically isomorphic to any matrix algebras over the real number field $\mathbb{R}$, because $\mathbb{O}$ is a non-associative algebra over $\mathbb{R}$. However since $\mathbb{O}$ is an extension of $\mathbb{H}$ by the Cayley-Dickson process and is also finite-dimensional, some pseudo real matrix representations of octonions can still be introduced through real matrix representations of quaternions. In this paper we give a complete investigation to real matrix representations of octonions, and consider their various applications to octonions as well as matrices of octonions.

AMS Mathematics Subject Classification: 15A33; 15A06; 15A24; 17A35

Key words: quaternions, octonions, matrix representations, linear equations, similarity, eigenvalues, Cayley-Hamilton theorem

1. Introduction

Let $\mathbb{O}$ be the octonion algebra over the real number field $\mathbb{R}$. Then it is well known by the Cayley-Dickson process that any $a \in \mathbb{O}$ can be written as

$$a = a' + a''e, \tag{1.1}$$

where a', $a'' \in \mathbb{H} = \{\, a = a_0 + a_1 i + a_2 j + a_3 k \mid i^2 = j^2 = k^2 = -1,\ ijk = -1,\ a_0\text{—}a_3 \in \mathbb{R}\,\}$, the real quaternion division algebra. The addition and multiplication for any $a = a' + a''e,\ b = b' + b''e \in \mathbb{O}$ are defined by

$$a + b = (\,a' + a''e\,) + (\,b' + b''e) = (\,a' + b'\,) + (\,a'' + b''\,)e, \tag{1.2}$$

and

$$ab = (\,a' + a''e)(b' + b''e) = (a'b' - \overline{b''}a'') + (b''a' + a''\overline{b'}\,)e, \tag{1.3}$$

where $\overline{b'}$, $\overline{b''}$ denote the conjugates of the quaternions b' and b''. In that case, $\mathbb{O}$ is an eight-dimensional non-associative but alternative division algebra over its center field $\mathbb{R}$, and the canonical basis of $\mathbb{O}$ is

$$1, \quad e_1 = i, \quad e_2 = j, \quad e_3 = k, \quad e_4 = e, \quad e_5 = ie, \quad e_6 = je, \quad e_7 = ke. \tag{1.4}$$

The multiplication rules for the basis of $\mathbb{O}$ are listed in the following matrix

$$E_8^T E_8 = \begin{bmatrix} 1 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ e_1 & -1 & e_3 & -e_2 & e_5 & -e_4 & -e_7 & e_6 \\ e_2 & -e_3 & -1 & e_1 & e_6 & e_7 & -e_4 & -e_5 \\ e_3 & e_2 & -e_1 & -1 & e_7 & -e_6 & e_5 & -e_4 \\ e_4 & -e_5 & -e_6 & -e_7 & -1 & e_1 & e_2 & e_3 \\ e_5 & e_4 & -e_7 & e_6 & -e_1 & -1 & -e_3 & e_2 \\ e_6 & e_7 & e_4 & -e_5 & -e_2 & e_3 & -1 & -e_1 \\ e_7 & -e_6 & e_5 & e_4 & -e_3 & -e_2 & e_1 & -1 \end{bmatrix}, \tag{1.5}$$

where $E_8 = [\,1,\ e_1,\ \cdots,\ e_7\,]$. Under Eq.(1.4) all elements of $\mathbb{O}$ take the form

$$a = a_0 + a_1e_1 + \cdots + a_7e_7, \tag{1.6}$$

where a_0—$a_7 \in \mathbb{R}$, which can also simply be written as $a = \operatorname{Re} a + \operatorname{Im} a$, where $\operatorname{Re} a = a_0$. The conjugate of a is defined to be

$$\overline{a} = \overline{a'} - a''e = \operatorname{Re} a - \operatorname{Im} a. \tag{1.7}$$

This operation satisfies

$$\overline{\overline{a}} = a, \quad \overline{a + b} = \overline{a} + \overline{b}, \quad \overline{ab} = \overline{b}\,\overline{a} \tag{1.8}$$

for all $a, b \in \mathbb{O}$. The norm of a is defined to be $|a| := \sqrt{a\overline{a}} = \sqrt{\overline{a}a} = \sqrt{a_0^2 + a_1^2 + \cdots + a_7^2}$. Although $\mathbb{O}$ is nonassociative, it is still an alternative, flexible, quadratic, composition and division algebra over $\mathbb{R}$, that is, for all

$a,\ b \in \mathbb{O}$, the following equalities hold:

$$a(ab) = a^2b, \qquad (ba)a = ba^2, \qquad (ab)a = a(ba) := aba, \tag{1.9}$$

$$a^{-1} = \frac{\overline{a}}{|a|^2}, \tag{1.10}$$

$$a^2 - 2(\operatorname{Re} a)a + |a|^2 = 0, \qquad (\operatorname{Im} a)^2 = -|\operatorname{Im} a|^2, \tag{1.11}$$

$$|ab| = |a||b|. \tag{1.12}$$

As is well known, any finite-dimensional associative algebra over an arbitrary field $\mathbb{F}$ is algebraically isomorphic to a subalgebra of a total matrix algebra over the field. In other words, any element in a finite-dimensional associative algebra over $\mathbb{F}$ has a faithful matrix representation over the field. For the real quaternion algebra $\mathbb{H}$, it is well known that through the bijective map

$$\phi : a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H} \longrightarrow \phi(a) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix}, \tag{1.13}$$

$\mathbb{H}$ is algebraically isomorphic to the matrix algebra

$$\mathcal{M} = \left\{ \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \,\middle|\, a_0,\ a_1,\ a_2,\ a_3 \in \mathbb{R} \right\}, \tag{1.14}$$

and $\phi(a)$ is a faithful real matrix representation of a. Our consideration for matrix representations of octonions are based on Eqs.(1.1)—(1.3) and the result in Eq.(1.13).

We next present some basic results related to matrix representations of quaternions, which will be serve as a tool for our examination in the sequel.

Lemma 1.1[13]. *Let* $a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H}$ *be given, where* a_0—$a_3 \in \mathbb{R}$. *Then the diagonal matrix* $\operatorname{diag}(\,a,\ a,\ a,\ a\,)$ *satisfies the following unitary similarity factorization equality*

$$Q \begin{bmatrix} a & & & \\ & a & & \\ & & a & \\ & & & a \end{bmatrix} Q^* = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \in \mathbb{R}^{4\times 4}, \tag{1.15}$$

where the matrix Q has the following independent expression

$$Q = Q^* = \frac{1}{2}\begin{bmatrix} 1 & i & j & k \\ -i & 1 & k & -j \\ -j & -k & 1 & i \\ -k & j & -i & 1 \end{bmatrix}, \tag{1.16}$$

which is a unitary matrix over $\mathbb{H}$.

Lemma 1.2[13]. *Let $a, b \in \mathbb{H}$, and $\lambda \in \mathbb{R}$. Then*

(a) $a = b \Longleftrightarrow \phi(a) = \phi(b)$.

(b) $\phi(a+b) = \phi(a)+\phi(b), \quad \phi(ab) = \phi(a)\phi(b), \quad \phi(\lambda a) = \lambda\phi(a), \quad \phi(1) = I_4$.

(c) $a = \frac{1}{4}E_4\phi(a)E_4^*$, *where* $E_4 := [\,1,\ i,\ j,\ k\,]$ *and* $E_4^* := [\,1,\ -i,\ -j,\ -k\,]^T$.

(d) $\phi(\overline{a}) = \phi^T(a)$.

(e) $\phi(a^{-1}) = \phi^{-1}(a)$, *if* $a \neq 0$.

(f) $\det[\phi(a)] = |a|^4$.

We can also introduce from Eq.(1.13) another real matrix representation of a as follows

$$\tau(a) := K\phi^T(a)K = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix}, \tag{1.17}$$

where $K = \mathrm{diag}(\,1,\ -1,\ -1,\ -1\,)$. Some basic operation properties on $\tau(a)$ are

$$\tau(a+b) = \tau(a)+\tau(b), \qquad \tau(ab) = \tau(b)\tau(a), \qquad \tau(\overline{a}) = \tau^T(a), \tag{1.18}$$

$$\det[\phi(a)] = |a|^4, \qquad \phi(a^{-1}) = \phi^{-1}(a) \quad \text{if} \quad a \neq 0. \tag{1.19}$$

Combining the two real matrix representations of quaternions with their real vector representations, we have the following important result.

Lemma 1.3. *Let $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$, and denote $\overrightarrow{x} = [\,x_0,\ x_1,\ x_2,\ x_3\,]^T$, called the vector representation of x. Then for all $a, b, x \in \mathbb{H}$, we have*

$$\overrightarrow{ax} = \phi(a)\overrightarrow{x}, \qquad \overrightarrow{xb} = \tau(b)\overrightarrow{x}, \qquad \overrightarrow{axb} = \phi(a)\tau(b)\overrightarrow{x} = \tau(b)\phi(a)\overrightarrow{x}, \tag{1.20}$$

and the equality

$$\phi(a)\tau(b) = \tau(b)\phi(a) \tag{1.21}$$

always holds.

Proof. Observe that

$$\overrightarrow{x} = \phi(x)\alpha_4^T, \qquad \overrightarrow{x} = \tau(x)\alpha_4^T, \qquad \alpha_4 = [\,1,\ 0,\ 0,\ 0\,].$$

We find by Lemma 1.1 and Eq.(1.2) that

$$\overrightarrow{ax} = \phi(ax)\alpha_4^T = \phi(a)\phi(x)\alpha_4^T = \phi(a)\overrightarrow{x}, \qquad \overrightarrow{xb} = \tau(xb)\alpha_4^T = \tau(b)\tau(x)\alpha_4^T = \tau(b)\overrightarrow{x},$$

and

$$\overrightarrow{axb} = \overrightarrow{a(xb)} = \phi(a)\overrightarrow{(xb)} = \phi(a)\tau(b)\overrightarrow{x}, \quad \overrightarrow{axb} = \overrightarrow{(ax)b} = \tau(b)\overrightarrow{(ax)} = \tau(b)\phi(a)\overrightarrow{x}.$$

These four equalities are exactly the results in Eqs.(1.20) and (1.21). □

Lemma 1.4[12][17]. *Let $a,\ b,\ x \in \mathbb{O}$ be given. Then*

(a) $\mathrm{Re}\,(ab) = \mathrm{Re}\,(ba), \qquad \mathrm{Re}\,((ax)b) = \mathrm{Re}\,(a(xb))$.

(b) $(aba)x = a(b(ax)), \qquad x(aba) = ((xa)b)a$.

(c) $(ab)(xa) = a(bx)a, \qquad (bx)(ab) = b(xa)b$.

(d) $(a,\ b,\ x) = -(a,\ x,\ b) = (x,\ a,\ b)$, *where* $(a,\ b,\ x) = (ab)x - a(bx)$.

2. The Real Matrix Representations of Octonions

Based on the results on the real matrix representation of quaternions, we now can introduce real matrix representation of octonions.

Definition 2.1. Let $a = a' + a''e \in \mathbb{O}$, where $a' = a_0 + a_1 i + a_2 j + a_3 k$, $a'' = a_4 + a_5 i + a_6 j + a_7 k \in \mathbb{H}$. Then the 8×8 real matrix

$$\omega(a) := \begin{bmatrix} \phi(a') & -\tau(a'')K_4 \\ \phi(a'')K_4 & \tau(a') \end{bmatrix}, \tag{2.1}$$

is called the left matrix representation of a over $\mathbb{R}$, where $K_4 = \mathrm{diag}(1,\ -1,\ -1,\ -1)$. Written in an explicit form,

$$\omega(a) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & -a_3 & a_2 & -a_5 & a_4 & a_7 & -a_6 \\ a_2 & a_3 & a_0 & -a_1 & -a_6 & -a_7 & a_4 & a_5 \\ a_3 & -a_2 & a_1 & a_0 & -a_7 & a_6 & -a_5 & a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ a_5 & -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & -a_2 \\ a_6 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & a_0 & a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{bmatrix}, \tag{2.2}$$

Theorem 2.1. *Let $x = x_0 + x_1e_1 + \cdots + x_7e_7 \in \mathbb{O}$, and denote $\overrightarrow{x} = [x_0,\ x_1,\ \cdots,\ x_7]^T$, called the vector representation of x. Then*

$$\overrightarrow{ax} = \omega(a)\overrightarrow{x} \tag{2.3}$$

holds for $a,\ x \in \mathbb{O}$.

Proof. Write $a,\, x \in \mathbb{O}$ as $a = a' + a''e,\ x = x' + x''e$, where $a',\ a'',\ x',\ x'' \in \mathbb{H}$. We know by Eq.(1.3) that $ax = (a'x' - \overline{x''}a'') + (x''a' + a''\overline{x'})e$. Thus it follows by Eq.(1.20) that

$$\overrightarrow{ax} = \begin{bmatrix} \overrightarrow{a'x' - \overline{x''}a''} \\ \overrightarrow{x''a' + a''\overline{x'}} \end{bmatrix} = \begin{bmatrix} \overrightarrow{a'x'} - \overrightarrow{\overline{x''}a''} \\ \overrightarrow{x''a'} + \overrightarrow{a''\overline{x'}} \end{bmatrix}$$
$$= \begin{bmatrix} \phi(a')\overrightarrow{x'} - \tau(a'')K_4\overrightarrow{x''} \\ \tau(a')\overrightarrow{x''} + \phi(a'')K_4\overrightarrow{x'} \end{bmatrix}$$
$$= \begin{bmatrix} \phi(a') & -\tau(a'')K_4 \\ \phi(a'')K_4 & \tau(a') \end{bmatrix} \begin{bmatrix} \overrightarrow{x'} \\ \overrightarrow{x''} \end{bmatrix},$$

as required for Eq.(2.3). $\square$

Theorem 2.2. *Let $a \in \mathbb{O}$ be given. Then*

$$aE_8 = E_8\omega(a), \quad \text{and} \quad E_8^*a = \omega(a)E_8^*, \tag{2.4}$$

where $E_8 := [\,1,\ e_1,\ \cdots,\ e_7\,]$, and $E_8^* := [\,1,\ -e_1,\ \cdots,\ -e_7\,]^T$.

Proof. Follows from a direct verification. $\square$

We can also introduce from Eq.(2.1) another matrix representation for an octonion as follows.

Definition 2.2. Let $a = a' + a''e = a_0 + a_1e_1 + \cdots + a_7e_7 \in \mathbb{O}$ be given, where $a',\ a'' \in \mathbb{H}$. Then we call the 8×8 real matrix

$$\nu(a) := K_8\omega^T(a)K_8 = \begin{bmatrix} \tau(a') & -\phi(\overline{a''}) \\ \phi(a'') & \tau(\overline{a'}) \end{bmatrix}, \tag{2.5}$$

the right matrix representation of a, where $K_8 = \mathrm{diag}(\, K_4,\ I_4\,)$, an orthogonal matrix. Written in an explicit form,

$$\nu(a) = \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & a_3 & -a_2 & a_5 & -a_4 & -a_7 & a_6 \\ a_2 & -a_3 & a_0 & a_1 & a_6 & a_7 & -a_4 & -a_5 \\ a_3 & a_2 & -a_1 & a_0 & a_7 & -a_6 & a_5 & -a_4 \\ a_4 & -a_5 & -a_6 & -a_7 & a_0 & a_1 & a_2 & a_3 \\ a_5 & a_4 & -a_7 & a_6 & -a_1 & a_0 & -a_3 & a_2 \\ a_6 & a_7 & a_4 & -a_5 & -a_2 & a_3 & a_0 & -a_1 \\ a_7 & -a_6 & a_5 & a_4 & -a_3 & -a_2 & a_1 & a_0 \end{bmatrix}. \tag{2.6}$$

Theorem 2.3. *Let $a,\ x \in \mathbb{O}$ be given. Then*

$$\overrightarrow{xa} = \nu(a)\overrightarrow{x} \tag{2.7}$$

holds.

Proof. Write $a,\ x \in \mathbb{O}$ as $a = a' + a''e$, $x = x' + x''e$, where $a',\ a'',\ x',\ x'' \in \mathbb{H}$. We know by (1.3) that $xa = (\, x'a' - \overline{a''}x''\,) + (\, a''x' + x''\overline{a'}\,)e$. Thus we find by Eq.(1.20) that

$$\begin{aligned} \overrightarrow{xa} &= \begin{bmatrix} \overrightarrow{x'a' - \overline{a''}x''} \\ \overrightarrow{a''x' + x''\overline{a'}} \end{bmatrix} = \begin{bmatrix} \overrightarrow{x'a'} - \overrightarrow{\overline{a''}x''} \\ \overrightarrow{a''x'} + \overrightarrow{x''\overline{a'}} \end{bmatrix} = \\ &= \begin{bmatrix} \tau(a')\overrightarrow{x'} - \phi(\overline{a''})\overrightarrow{x''} \\ \phi(a'')\overrightarrow{x'} + \tau(\overline{a'})\overrightarrow{x''} \end{bmatrix} = \begin{bmatrix} \tau(a') & -\phi(\overline{a''}) \\ \phi(a'') & \tau(\overline{a'}) \end{bmatrix} \begin{bmatrix} \overrightarrow{x'} \\ \overrightarrow{x''} \end{bmatrix}, \end{aligned}$$

as required for Eq.(2.7). $\square$

Theorem 2.4. *Let $a \in \mathbb{O}$ be given. Then*

$$aF_8 = F_8\nu^T(a), \quad \textit{and} \quad F_8^* a = \nu^T(a)F_8^*, \tag{2.8}$$

where $F_8 := [\, 1,\ -e_1,\ \cdots,\ -e_7\,]$ and $F_8^* := [\, 1,\ e_1,\ \cdots,\ e_7\,]^T$.

Proof. Follows from a direct verification. $\square$

Observe from Eqs.(2.1) and (2.5) that the two real matrix representations of an octonion $a = a' + a''e$ are in fact constructed by the real matrix representations of two quaternions a' and a''. Hence the operation properties for the two matrix representations of octonions can easily be established through the results in Lemmas 1.2 and 1.3.

Theorem 2.5. *Let* $a,\ b \in \mathbb{O},\ \lambda \in \mathbb{R}$ *be given. Then*

(a) $a = b \Longleftrightarrow \omega(a) = \omega(b)$.

(b) $\omega(a+b) = \omega(a) + \omega(b), \qquad \omega(\lambda a) = \lambda\omega(a), \qquad \omega(1) = I_8$.

(c) $\omega(\overline{a}) = \omega^T(a)$.

Proof. Follows from a direct verification. □

Theorem 2.6. *Let* $a,\, b \in \mathbb{O},\, \lambda \in \mathbb{R}$ *be given. Then*

(a) $a = b \Longleftrightarrow \nu(a) = \nu(b)$.

(b) $\nu(a+b) = \nu(a) + \nu(b), \qquad \nu(\lambda a) = \lambda\nu(a), \qquad \nu(1) = I_8$.

(c) $\nu(\overline{a}) = \nu^T(a)$.

Proof. Follows from a direct verification. □

Theorem 2.7. *Let* $a \in \mathbb{O}$ *be given. Then*

$$a = \frac{1}{8}E_8\omega(a)E_8^*, \quad \textit{and} \quad a = \frac{1}{8}F_8\nu^T(a)F_8^*, \tag{2.9}$$

where $E_8,\ E_8^*,\ F_8$ *and* F_8^* *are as in Eqs.*(2.4) *and* (2.8).

Proof. Note that $\omega(a)$ and $\nu(a)$ are real matrices. Thus we get from Eqs.(2.4) and (2.8) that

$$E_8(E_8^*a) = E_8[\omega(a)E_8^*] = E_8\omega(a)E_8^*, \quad \text{and}$$

$$F(F_8^*a) = F_8[\nu^T(a)F_8^*] = F_8\nu^T(a)F_8^*.$$

On the other hand, note that $\mathbb{O}$ is alternative. It follows that

$$E_8(E_8^*a) = a - e_1(e_1a) - \cdots - e_7(e_7a) = a - e_1^2a - \cdots - e_7^2a = 8a,$$

and

$$F_8(F_8^*a) = a - e_1(e_1a) - \cdots - e_7(e_7a) = a - e_1^2a - \cdots - e_7^2a = 8a.$$

Thus we have Eq.(2.9). □

Theorem 2.8. *Let* $a \in \mathbb{O}$ *be given. Then*

$$\det[\omega(a)] = \det[\nu(a)] = |a|^8. \tag{2.10}$$

Proof. Write $a = a' + a''e$. Then we easily find by Eqs.(1.21) and (2.5) that

$$\begin{aligned} \det[\omega(a)] = \det[\nu(a)] = \begin{vmatrix} \tau(a') & -\phi(\overline{a''}) \\ \phi(a'') & \tau(\overline{a'}) \end{vmatrix} &= \det[\,\tau(a')\tau(\overline{a'}) + \phi(a'')\phi(\overline{a''})\,] \\ &= \det[\,\tau(\overline{a'}a') + \phi(a''\overline{a''})\,] \\ &= \det[\,|a'|^2 I_4 + |a''|^2 I_4\,] \\ &= (\,|a'|^2 + |a''|^2\,)^4 = |a|^8, \end{aligned}$$

as required for Eq.(2.10). $\square$

Theorem 2.9. *Let $a \in \mathbb{O}$ be given. Then the two matrix representations of a satisfy the following three identities*

$$\omega(a^2) = \omega^2(a), \qquad \nu(a^2) = \nu^2(a), \qquad \omega(a)\nu(a) = \nu(a)\omega(a). \tag{2.11}$$

Proof. Applying Eqs.(2.3) and (2.7) to both sides of the three identities in Eq.(1.9) leads to

$$\omega^2(a)\overrightarrow{b} = \omega(a^2)\overrightarrow{b}, \quad \nu^2(a)\overrightarrow{b} = \nu(a^2)\overrightarrow{b}, \quad \omega(a)\nu(a)\overrightarrow{b} = \nu(a)\omega(a)\overrightarrow{b}.$$

Note that $\overrightarrow{b}$ is an arbitrary 8×1 real vector when b runs over $\mathbb{O}$. Thus Eq.(2.11) follows. $\square$

Theorem 2.10. *Let $a \in \mathbb{O}$ be given with $a \neq 0$. Then*

$$\omega(a^{-1}) = \omega^{-1}(a), \quad \textit{and} \quad \nu(a^{-1}) = \nu^{-1}(a). \tag{2.12}$$

Proof. Note from Eqs.(1.10) and (1.11) that

$$a^{-1} = \frac{\overline{a}}{|a|^2} = \frac{1}{|a|^2}[\,2(\mathrm{Re}\,a) - a\,]$$

and

$$a^2 - 2\mathrm{Re}\,a + |a|^2 = 0.$$

Applying Theorems 2.5 and 2.6, as well as the first two equalities in Eq.(2.11) to both sides of the above two equalities, we obtain

$$\omega(a^{-1}) = \frac{1}{|a|^2}[\,2(\mathrm{Re}\,a)I_8 - \omega(a)\,], \qquad \nu(a^{-1}) = \frac{1}{|a|^2}[\,2(\mathrm{Re}\,a)I_8 - \nu(a)\,]$$

and

$$\omega^2(a) - 2(\mathrm{Re}\, a)\omega(a) + |a|^2 I_8 = 0, \qquad \nu^2(a) - 2(\mathrm{Re}\, a)\nu(a) + |a|^2 I_8 = 0.$$

Contrasting them yields Eq.(2.12). $\square$

Because $\mathbb{O}$ is non-associative, the operation properties $\omega(ab) = \omega(a)\omega(b)$ and $\nu(ab) = \nu(b)\nu(a)$ do not hold in general, otherwise $\mathbb{O}$ will be algebraically isomorphic to or algebraically anti-isomorphic to an associative matrix algebra over $\mathbb{R}$, this is impossible. Nevertheless, some other kinds of identities on the two real matrix representations of octonions can still be established from the identities in Lemma 1.4(a)—(d).

Theorem 2.11. *Let $a,\, b \in \mathbb{O}$ be given. Then their matrix representations satisfy the following two identities*

$$\omega(aba) = \omega(a)\omega(b)\omega(a), \quad \textit{and} \quad \nu(aba) = \nu(a)\nu(b)\nu(a). \tag{2.13}$$

Proof. Follows from applying Eqs.(2.3) and (2.7) to the Moufang identities in Lemma 1.4(b) . $\square$

Theorem 2.12. *Let $a,\, b \in \mathbb{O}$ be given. Then their matrix representations satisfy the following identities*

$$\omega(ab) + \omega(ba) = \omega(a)\omega(b) + \omega(b)\omega(a), \tag{2.14}$$
$$\nu(ab) + \nu(ba) = \nu(a)\nu(b) + \nu(b)\nu(a), \tag{2.15}$$
$$\omega(ab) + \nu(ab) = \omega(a)\omega(b) + \nu(b)\nu(a), \tag{2.16}$$
$$\omega(a)\nu(b) + \omega(b)\nu(a) = \nu(a)\omega(b) + \nu(b)\omega(a), \tag{2.17}$$
$$\omega(ab) = \omega(a)\omega(b) + \omega(a)\nu(b) - \nu(b)\omega(a), \tag{2.18}$$
$$\nu(ab) = \nu(b)\nu(a) + \omega(b)\nu(a) - \nu(a)\omega(b). \tag{2.19}$$

Proof. The identities in Lemma 1.4(d) can clearly be written as the following six identities

$$(ab)x - a(bx) = -(ba)x + b(ax), \qquad (xa)b - x(ab) = -(xb)a + x(ba),$$

$$(ab)x - a(bx) = -(bx)a + b(xa), \qquad (ab)x - a(bx) = -(xa)b + x(ab),$$

$$(ab)x - a(bx) = -(ax)b + a(xb), \qquad (xa)b - x(ab) = -(ax)b + a(xb).$$

Applying Eqs.(2.3) and (2.7) to both sides of the above identities, we obtain

$$\begin{aligned}
&[\,\omega(ab) - \omega(a)\omega(b)\,]\overrightarrow{x} = [\,-\omega(ba) + \omega(b)\omega(a)\,]\overrightarrow{x},\\
&[\,\nu(b)\nu(a) - \nu(ab)\,]\overrightarrow{x} = [\,-\nu(a)\nu(b) + \nu(ba)\,]\overrightarrow{x},\\
&[\,\nu(b)\omega(a) - \omega(a)\nu(b)\,]\overrightarrow{x} = [\,-\nu(a)\omega(b) + \omega(b)\nu(a)\,]\overrightarrow{x},\\
&[\,\omega(ab) - \omega(a)\omega(b)\,]\overrightarrow{x} = [\,-\nu(b)\nu(a) + \nu(ab)\,]\overrightarrow{x},\\
&[\,\omega(ab) - \omega(a)\omega(b)\,]\overrightarrow{x} = [\,-\nu(b)\omega(a) + \omega(a)\nu(b)\,]\overrightarrow{x},\\
&[\,\nu(b)\nu(a) - \nu(ab)\,]\overrightarrow{x} = [\,-\nu(b)\omega(a) + \omega(a)\nu(b)\,]\overrightarrow{x}.
\end{aligned}$$

Notice that $\overrightarrow{x}$ is an arbitrary real 8×1 real matrix when x runs over $\mathbb{O}$. Therefore Eqs.(2.14)—(2.19) follow. □

Theorem 2.13. *Let $a,\, b \in \mathbb{O}$ be given with $a \neq 0,\, b \neq 0$. Then their matrix representations satisfy the following two identities*

$$\omega(ab) = \nu(a)[\,\omega(a)\omega(b)\,]\nu^{-1}(a), \quad \textit{and} \quad \nu(ab) = \omega(b)[\,\nu(b)\nu(a)\,]\omega^{-1}(b). \tag{2.20}$$

which imply that

$$\omega(ab) \sim \omega(a)\omega(b), \quad \textit{and} \quad \nu(ab) \sim \nu(b)\nu(a). \tag{2.21}$$

Proof. Applying Eqs.(2.3) and (2.7) to the both sides of the two identities in Lemma 1.4(c), we obtain

$$\omega(ab)\nu(a)\overrightarrow{x} = \nu(a)\omega(a)\omega(b)\overrightarrow{x}, \quad \text{and} \quad \nu(ab)\omega(b)\overrightarrow{x} = \omega(b)\nu(b)\nu(a)\overrightarrow{x},$$

which are obviously equivalent to Eq.(2.20). □

Note from Eqs.(2.3) and (2.7) that any linear equation of the form $ax - xb = c$ over $\mathbb{O}$ can equivalently be written as $[\,\omega(a) - \nu(b)\,]\overrightarrow{x} = \overrightarrow{a}$, which is a linear equation over $\mathbb{R}$. Thus it is necessary to consider the operation properties of the matrix $\omega(a) - \nu(b)$, especially the determinant of $\omega(a) - \nu(b)$ for any $a,\ b \in \mathbb{O}$. Here we only list the expression of the determinant of $\omega(a) - \nu(b)$. Its proof is quite tedious and is, therefore, omitted here.

Theorem 2.14. *Let $a,\, b \in \mathbb{O}$ be given and define $\delta(a,\ b) := \omega(a) - \nu(b)$. Then*

$$\det[\delta(a,\ b)] = |a - \overline{b}|^4[\,s^2 + (\,|\mathrm{Im}\,a| - |\mathrm{Im}\,b|\,)^2\,][\,s^2 + (\,|\mathrm{Im}\,a| + |\mathrm{Im}\,b|\,)^2\,] \tag{2.22}$$

$$\det[\delta(a,\ b)] = (\,s^2 + |\mathrm{Im}\,a + \mathrm{Im}\,b|^2\,)^2$$
$$[s^4 + 2s^2(|\mathrm{Im}\,a|^2 + |\mathrm{Im}\,b|^2) + (\,|\mathrm{Im}\,a|^2 - |\mathrm{Im}\,b|^2\,)^2], \tag{2.23}$$

where $s = \mathrm{Re}\, a - \mathrm{Re}\, b$*. The characteristic polynomial of* $\delta(a,\ b)$ *is*

$$
\begin{aligned}
&|\, \lambda I_8 - \delta(a,\ b)\, | \\
&= [\, (\lambda - s\,)^2 + |\mathrm{Im}\, a + \mathrm{Im}\, b|^2]^2 [\, (\lambda - s\,)^2 + \\
&+ (|\mathrm{Im}\, a| - |\mathrm{Im}\, b|\,)^2\,][\, (\lambda - s\,)^2 + (\, |\mathrm{Im}\, a| + |\mathrm{Im}\, b|\,)^2\,].
\end{aligned} \tag{2.24}
$$

In particular, if $\mathrm{Re}\, a = \mathrm{Re}\, b$ *and* $|\mathrm{Im}\, a| = |\mathrm{Im}\, b|$*, but* $a \neq \overline{b}$*, then*

$$
\mathrm{rank}\, \delta(a,\, b) = 6. \tag{2.25}
$$

Theorem 2.15. *Let* $a, b \in \mathbb{O}$ *be given with* $a \neq 0$ *and* $b \neq 0$*. Then* $\delta(a,\ b) = \omega(a) - \nu(b)$ *is a real normal matrix over* $\mathbb{R}$*, that is,* $\delta(a,\ b)\delta^T(a,\ b) = \delta^T(a,\ b)\delta(a,\ b)$*.*

Proof. Follows from

$$
\begin{aligned}
\delta(a,\ b) + \delta^T(a,\ b) &= \omega(a) - \nu(b) + \omega^T(a) - \nu^T(b) \\
&= \omega(a) - \nu(b) + \omega(\overline{a}) - \nu(\overline{b}) \\
&= \omega(a + \overline{a}) - \nu(b + \overline{b}) = 2(\mathrm{Re}\, a - \mathrm{Re}\, b) I_8. \qquad \square
\end{aligned}
$$

Theorem 2.16. *Let* $a \in \mathbb{O}$ *be given with* $a \notin \mathbb{R}$*. Then*

$$
\delta^3(a,\ a) = -4|\mathrm{Im}\, a|^2 \delta(a,\ a), \tag{2.26}
$$

and $\delta(a,\ a)$ *has a generalized inverse as follows*

$$
\delta^-(a,\ a) = -\frac{1}{4|\mathrm{Im}\, a|^2}\delta(a,\ a). \tag{2.27}
$$

Proof. Observe that $\delta(a,\ a) = \omega(a) - \nu(a) = \omega(\mathrm{Im}\, a) - \nu(\mathrm{Im}\, a)$ and $(\mathrm{Im}\, a)^2 = -|\mathrm{Im}\, a|^2$. Thus we find that

$$
\begin{aligned}
\delta^2(a,\ a) &= [\, \omega(\mathrm{Im}\, a) - \nu(\mathrm{Im}\, a)\,]^2 \\
&= [\, \omega^2(\mathrm{Im}\, a) - 2\omega(\mathrm{Im}\, a)\nu(\mathrm{Im}\, a) + \nu^2(\mathrm{Im}\, a)\,] \\
&= [\, \omega((\mathrm{Im}\, a)^2) - 2\omega(\mathrm{Im}\, a)\nu(\mathrm{Im}\, a) + \nu((\mathrm{Im}\, a)^2)\,] \\
&= -2[\, |\mathrm{Im}\, a|^2 I_8 + \omega(\mathrm{Im}\, a)\nu(\mathrm{Im}\, a)\,],
\end{aligned}
$$

and

$$
\begin{aligned}
\delta^3(a,\ a) &= -2[\, |\mathrm{Im}\, a|^2 I_8 + \omega(\mathrm{Im}\, a)\nu(\mathrm{Im}\, a)\,][\, \omega(\mathrm{Im}\, a) - \nu(\mathrm{Im}\, a)\,] \\
&= -4|\mathrm{Im}\, a|^2[\, \omega(\mathrm{Im}\, a) - \nu(\mathrm{Im}\, a)\,] = -4|\mathrm{Im}{,}a|^2 \delta(a,\ a),
\end{aligned}
$$

as required for Eq.(2.26). $\square$

3. Some Linear Equations Over $\mathbb{O}$

The matrix expressions of octonions and their properties introduced in Section 2 enable us to easily deal with various problems related to octonions. One of the most fundamental topics on octonions is concerning solutions of various linear equations over $\mathbb{O}$. In this section, we shall give a complete discussion for this problem. Our first result is concerning the linear equation $ax = xb$, which was examined by the author in [14].

Thoerem 3.1[14]. *Let $a = a_0 + a_1e_1 + \cdots + a_7e_7$, $b = b_0 + b_1e_1 + \cdots + b_7e_7 \in \mathbb{O}$ be given. Then the linear equation $ax = xb$ has a nonzero solution if and only if*

$$\operatorname{Re} a = \operatorname{Re} b \quad \textit{and} \quad |\operatorname{Im} a| = |\operatorname{Im} b|. \tag{3.1}$$

(a) *In that case, if $b \neq \overline{a}$, i. e.,* $\operatorname{Im} a + \operatorname{Im} b \neq 0$, *then the general solution of $ax = xb$ can be expressed as*

$$x = (\operatorname{Im} a)p + p(\operatorname{Im} b), \tag{3.1}$$

where $p \in \mathcal{A}(a,\, b)$, the subalgebra generated by a and b, is arbitrary or equivalently

$$x = \lambda_1(\operatorname{Im} a + \operatorname{Im} b) + \lambda_2[\,|\operatorname{Im} a|\,|\operatorname{Im} b| - (\operatorname{Im} a)(\operatorname{Im} b)\,], \tag{3.2}$$

where $\lambda_1,\, \lambda_2 \in \mathbb{R}$ are arbitrary.

(b) *If $b = \overline{a}$, then the general solution of $ax = xb$ is*

$$x = x_1e_1 + x_2e_2 + \cdots + x_7e_7, \tag{3.3}$$

where x_1—x_7 satisfy $a_1x_1 + a_2x_2 + \cdots + a_7x_7 = 0$.

The correctness of this result can be directly verified by substitution.

Based on the equation $ax = xb$, we can define the similarity of two octonions. Two octonions are said to be *similar* if there is a nonzero $p \in \mathbb{O}$ such that $a = pbp^{-1}$, which is written as $a \sim b$. Theorem 3.1 shows that two octonions are similar if and only if $\operatorname{Re} a = \operatorname{Re} b$ and $|\operatorname{Im} a| = |\operatorname{Im} b|$. Thus the similarity defined here is also an equivalence relation on octonions. In addition, we have the following.

Theorem 3.2. *Let $a,\, b \in \mathbb{O}$ be given with $b \neq \overline{a}$. Then*

$$a \sim b \Longleftrightarrow \omega(a) \sim \omega(b). \tag{3.4}$$

Proof. Suppose first that $a \sim b$. Then it follows by Eq.(1.11) that

$$a^2 - 2(\mathrm{Re}\, a)a = -|a|^2 = -|b|^2 = b^2 - 2(\mathrm{Re}\, b)b.$$

Applying Theorem 2.5(a) and Eq.(2.11) to both sides of the above equality and we get

$$\omega^2(a) - 2(\mathrm{Re}\, a)\omega(a) = \omega^2(b) - 2(\mathrm{Re}\, b)\omega(b).$$

Thus

$$\omega^2(a) + \omega(a)\omega(b) - 2(\mathrm{Re}\, a)\omega(a) = \omega^2(b) + \omega(a)\omega(b) - 2(\mathrm{Re}\, b)\omega(b),$$

which is equivalent to

$$\omega(a)[\,\omega(a) + \omega(b) - 2(\mathrm{Re}\, a)I_8\,] = [\,\omega(a) + \omega(b) - 2(\mathrm{Re}\, b)I_8\,]\omega(b),$$

or simply

$$\omega(a)\omega(\mathrm{Im}\, a + \mathrm{Im}\, b) = \omega(\mathrm{Im}\, a + \mathrm{Im}\, b)\omega(b).$$

Note that $\mathrm{Im}\, a + \mathrm{Im}\, b \neq 0$. Thus $\omega(\mathrm{Im}\, a + \mathrm{Im}\, b)$ is invertible. The above equality shows that $\omega(a) \sim \omega(b)$. Conversely, if $\omega(a) \sim \omega(b)$, then $\mathrm{trace}\,\omega(a) = \mathrm{trace}\,\omega(b)$ and $|\omega(a)| = |\omega(b)|$, which are equivalent to Eq.(3.1). $\square$

Next we consider some nonhomogeneous linear equations over $\mathbb{O}$.

Theorem 3.3. *Let $a, b \in \mathbb{O}$ be given with $a \notin \mathbb{R}$. Then the linear equation $ax - xa = b$ has a solution in $\mathbb{O}$ if and only if the equality $ab = b\overline{a}$ holds. In this case, the general solution of $ax - xa = b$ is*

$$x = \frac{1}{4|\mathrm{Im}\, a|^2}(\,ba - ab\,) + p - \frac{1}{|\mathrm{Im}\, a|^2}(\mathrm{Im}\, a)p(\mathrm{Im}\, a), \tag{3.5}$$

where $p \in \mathbb{O}$ is arbitrary.

Proof. According to Eqs.(2.3) and (2.7), the equation $ax - xa = b$ can equivalently be written as

$$[\,\omega(a) - \nu(a)\,]\overrightarrow{x} = \delta(a,\ a)\overrightarrow{x} = \overrightarrow{b}. \tag{3.6}$$

This equation is solvable if and only if $\delta(a,\ a)\delta^{-}(a,\ a)\overrightarrow{b} = \overrightarrow{b}$. In that case, the general solution of Eq.(3.6) can be expressed as

$$\overrightarrow{x} = \delta^{-}(a,\ a)\overrightarrow{c} + 2[\,I_8 - \delta^{-}(a,\ a)\delta(a,\ a)\,]\overrightarrow{p},$$

where $\overrightarrow{p}$ is an arbitrary real vector. Substituting

$$\delta^{-}(a,\ a) = -\frac{1}{4|\mathrm{Im}\,a|^2}\delta(a,\ a), \quad \textit{and} \quad \delta^2(a,\ a) = -2[\,|\mathrm{Im}\,a|^2 + \omega(\mathrm{Im}\,a)\nu(\mathrm{Im}\,a)\,]$$

in the above two equalities and then returning them to octonion forms by Eqs.(2.3) and (2.7) produce the equality in Part (b) and Eq.(3.5). □

Theorem 3.4. *Let $a = a_0 + a_1e_1 + \cdots + a_7e_7$, $b = b_0 + b_1e_1 + \cdots + b_7e_7 \in \mathbb{O}$ be given with $a \notin \mathbb{R}$. Then the equation*

$$ax - x\overline{a} = b \tag{3.7}$$

has a solution if and only if there exist $\lambda_0,\ \lambda_1 \in \mathbb{R}$ such that

$$b = \lambda_0 + \lambda_1 a, \tag{3.8}$$

in which case, the general solution of Eq.(3.7) *is*

$$x = \frac{\lambda_1}{2} + x_1e_1 + \cdots + x_7e_7, \tag{3.9}$$

where x_1—x_7 satisfy

$$a_1x_1 + \cdots + a_7x_7 = -\frac{1}{2}\mathrm{Re}\,b. \tag{3.10}$$

Proof. According to Eqs.(2.3) and (2.7), the equation (3.7) is equivalent to

$$[\,\omega(a) - \nu(\overline{a})\,]\overrightarrow{x} = \delta(a,\ \overline{a})\overrightarrow{x} = \overrightarrow{b}, \tag{3.11}$$

namely

$$\begin{bmatrix} 0 & -2a_1 & \cdots & -2a_7 \\ 2a_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 2a_7 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_7 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_7 \end{bmatrix}.$$

Obviously, this equation is solvable if and only if there is a $\lambda_1 \in \mathbb{R}$ such that

$$b_1 = \lambda_1 a_1, \quad b_2 = \lambda_1 a_2, \quad \cdots, \quad b_7 = \lambda_1 a_7,$$

i. e., $\mathrm{Im\,b} = \lambda_1 \mathrm{Im\,a}$, which is equivalent to Eq.(3.8). In that case, the solution to x_0 is $x_0 = \frac{\lambda_1}{2}$, and x_1—x_7 are determined by Eq.(3.9). □

Next we consider the linear equation

$$ax - xb = c \tag{3.12}$$

under the condition $a \sim b$. Clearly Eq.(3.12) is equivalent to

$$[\,\omega(a) - \nu(b)\,]\overrightarrow{x} = \delta(a,\ b)\overrightarrow{x} = \overrightarrow{c}. \tag{3.13}$$

Under $a \sim b$, we know by Theorem 3.3 that $ax = xb$ has a nonzero solution. Hence $\delta(a,\ b)$ is singular under $a \sim b$. In that case, Eq.(3.12) is solvable if and only if

$$\delta(a,\ b)\delta^{-}(a,\ b)\overrightarrow{c} = \overrightarrow{c}, \tag{3.14}$$

and the general solution of Eq.(3.13) is

$$\overrightarrow{x} = \delta^{-}(a,\ b)\overrightarrow{c} + 2[\,I_8 - \delta^{-}(a, b)\delta(a, b)\,]\overrightarrow{p}, \tag{3.15}$$

where $\overrightarrow{p}$ is an arbitrary real vector. If a is not similar to b. Clearly Eq.(3.13) has a unique solution

$$\overrightarrow{x} = \delta^{-1}(a,\ b)\overrightarrow{c} \tag{3.16}$$

Eqs.(3.15) and (3.16) show that the solvability and solution of the octonion equation (3.12) can be completely determined by its real adjoint linear system of equations (3.13). Through the characteristic polynomial (2.24), one can also return Eqs.(3.15) and (3.16) to octonion forms. But their expressions are quite tedious in form, and are omitted here.

Another instinctive linear equation over $\mathbb{O}$ is

$$a(xb) - (ax)b = c, \tag{3.17}$$

which is also equivalent to

$$(ab)x - a(bx) = c, \tag{3.18}$$

as well as

$$x(ab) - (xa)b = c, \tag{3.19}$$

because $(ab)x - a(bx) = (ab)x - a(bx) = x(ab) - (xa)b$ hold for all $a,\ b,\ x \in \mathbb{O}$. Now applying Eqs.(2.3) and (2.7) to both sides of Eq.(3.17), we obtain an equivalent equation

$$[\,\omega(a)\nu(b) - \nu(b)\omega(a)\,]\overrightarrow{x} = \overrightarrow{c}. \tag{3.20}$$

Here we set $\mu(a,\ b) = \omega(a)\nu(b) - \nu(b)\omega(a)$. Then it is easy to see that Eq.(3.20) is solvable if and only if

$$\mu(a,\ b)\mu^{-}(a,\, b)\overrightarrow{c} = \overrightarrow{c},$$

where $\mu^{-}(a,\, b)$ is a generalized inverse of $\mu(a,\, b)$. In that case, the general solution of Eq.(3.20) is

$$\overrightarrow{x} = \mu^{-}(a,\, b)\overrightarrow{c} + [\, I_8 - \mu^{-}(a,\, b)\mu(a,\, b)\,]\overrightarrow{p}, \tag{3.21}$$

where $\overrightarrow{p}$ is an arbitrary real vector. Numerical computation for Eq.(3.21) can reveal some interesting facts on Eq.(3.17). The reader can try to find them.

Theoretically speaking, any kind of two-sided linear equations or systems of linear equations over $\mathbb{O}$ can be equivalently transformed into systems of linear equations over $\mathbb{R}$ by the two equalities in Eqs.(2.3) and (2.7). Thus the problems related to linear equations over $\mathbb{O}$ now have a complete resolution.

4. Real Adjoint Matrices of Octonion Matrices

In this section, we consider how to extend the work in Sections 2 and 3 to octonion matrices and use them to deal with various octonion matrix problems. Since octonion algebra is non-associative, the matrix operations in $\mathbb{O}$ is much different from what we are familiar with in an associative algebra. Even the simplest matrix multiplication rule $A^2A = AA^2$ does not hold over $\mathbb{O}$, that is to say, multiplication of matrices over $\mathbb{O}$ is completely not associative. Thus nearly all the known results and methods on matrices over associative algebras can hardly be extended to matrices over $\mathbb{O}$. In that case, a unique method available to deal with matrices over $\mathbb{O}$ is to establish real matrix representations of octonion matrices, and then to transform matrix problems over $\mathbb{O}$ to various equivalent real matrix problems.

Based on the two matrix representations of octonions shown in Eqs.(2.2) and (2.6), we now introduce two adjoints for an octonion matrix as follows.

Definition 4.1. Let $A = (a_{st}) \in \mathbb{O}^{m\times n}$ be given. Then the *left adjoint matrix* of A is defined to be

$$\omega(A) = [\omega(a_{st})] = \begin{bmatrix} \omega(a_{11}) & \cdots & \omega(a_{1n}) \\ \vdots & & \vdots \\ \omega(a_{m1}) & \cdots & \omega(a_{mn}) \end{bmatrix} \in \mathbb{R}^{8m\times 8n}, \tag{4.1}$$

the *right adjoint matrix* of A is defined to be

$$\nu(A) = [\nu(a_{ts})] = \begin{bmatrix} \nu(a_{11}) & \cdots & \nu(a_{m1}) \\ \vdots & & \vdots \\ \nu(a_{1n}) & \cdots & \nu(a_{mn}) \end{bmatrix} \in \mathbb{R}^{8n\times 8m}, \tag{4.2}$$

and the *adjoint vector* of A is defined to be

$$\mathrm{vec}A := [\,\overrightarrow{a_{11}}^T,\ \cdots,\ \overrightarrow{a_{m1}}^T,\ \overrightarrow{a_{12}}^T,\ \cdots,\ \overrightarrow{a_{m2}}^T,\ \cdots,\ \overrightarrow{a_{1n}}^T,\ \cdots,\ \overrightarrow{a_{mn}}^T\,]^T. \tag{4.3}$$

Definition 4.2. Let $A = (A_{st})_{m\times n}$ and $B = (B_{st})_{p\times q}$ are two block matrices over $\mathbb{R}$, where $A_{st},\ B_{st} \in \mathbb{R}^{8\times 8}$. Then the *left and right block Kronecker products* of A and B, denoted respectively by $A\widehat{\otimes}B$ and $A\tilde{\otimes}B$, are defined to be

$$A\widehat{\otimes}B = \begin{bmatrix} A_{11}\odot_L B & \cdots & A_{1n}\odot_L B \\ \vdots & \ddots & \vdots \\ A_{m1}\odot_L B & \cdots & A_{mn}\odot_L B \end{bmatrix} \in \mathbb{R}^{8mp\times 8nq}, \tag{4.4}$$

and

$$A\tilde{\otimes}B = \begin{bmatrix} A\odot_R B_{11} & \cdots & A\odot_R B_{1q} \\ \vdots & \ddots & \vdots \\ A\odot_R B_{p1} & \cdots & A\odot_R B_{pq} \end{bmatrix} \in \mathbb{R}^{8mp\times 8nq}, \tag{4.5}$$

where

$$A_{st}\odot_L B = \begin{bmatrix} A_{st}B_{11} & \cdots & A_{st}B_{1q} \\ \vdots & \ddots & \vdots \\ A_{st}B_{p1} & \cdots & A_{st}B_{pq} \end{bmatrix} \in \mathbb{R}^{8p\times 8q}, \tag{4.6}$$

$$A\odot_R B_{st} = \begin{bmatrix} A_{11}B_{st} & \cdots & A_{1n}B_{st} \\ \vdots & \ddots & \vdots \\ A_{m1}B_{st} & \cdots & A_{mn}B_{st} \end{bmatrix} \in \mathbb{R}^{8m\times 8n}. \tag{4.7}$$

Noticing the equality (2.5), we see the two adjoint matrices $\omega(A)$ and $\nu(A)$ of an octonion matrix A satisfy the following equality

$$\nu(A) = K_{8n}\omega^T(A)K_{8m}, \tag{4.8}$$

where

$$K_{8t} = \mathrm{diag}(\,K_8,\ \cdots,\ K_8\,), \qquad K_8 = \mathrm{diag}(\,1,\ -1,\ \cdots,\ -1\,), \qquad t = m,\ n. \tag{4.9}$$

It is easy to see from Eqs.(4.4) and (4.5) that the two kinds of block Kronecker products are actually constructed by replacing all elements in the standard Kronecker product of matrices with 8×8 matrices. Hence the operation properties on these two kinds of products are much similar to those on the standard Kronecker product of matrices. We do not intend to list them here.

We next present some operation properties on the two real matrix representations of octonion matrices.

Theorem 4.1. *Let $A,\ B \in \mathbb{O}^{m\times n}$, $\lambda \in \mathbb{R}$ be given. Then*

(a) $A = B \Longleftrightarrow \omega(A) = \omega(B) \Longleftrightarrow \ \nu(A) = \nu(B)$, *i. e., ω and ν are 1-1.*

(b) $\omega(A+B) = \omega(A) + \omega(B)$, *and* $\nu(A+B) = \nu(A) + \nu(B)$.

(c) $\omega(\lambda A) = \lambda\omega(A)$, *and* $\nu(\lambda A) = \lambda\nu(A)$.

(d) $\omega(I_m) = I_{8m}$, *and* $\nu(I_m) = I_{8m}$.

(e) $\omega(A^*) = \omega^T(A)$, *and* $\nu(A^*) = \nu^T(A)$, *where* $A^* = (\overline{a_{ts}})$ *is the conjugate transpose of A.*

Theorem 4.2. *Let $A \in \mathbb{O}^{m\times n}$ be given. Then*

$$A = \frac{1}{8}E_{8m}\omega(A)E_{8n}^T, \tag{4.10}$$

where

$$E_{8t} = \mathrm{diag}(\, E_8,\ \cdots,\ E_8\,), \quad \textit{and} \quad E_8 = \mathrm{diag}(\,1,\ e_1,\ \cdots,\ e_7\,), \quad t = m,\ n.$$

Proof. Follows directly from Corollary 2.7. $\square$

Since the multiplication of matrices over $\mathbb{O}$ is completely not associative, no identities on products of octonions matrices can be established over $\mathbb{O}$ in general. Consequently, no identities on products of the two kinds of real matrix representations of octonion matrices can be established. In spite of this, we can still apply Eqs.(4.1) and (4.2) to deal with various problems related to octonion matrices. Next are some results on the relationship of $\omega(\cdot)$, $\nu(\cdot)$ and $\mathrm{vec}(\cdot)$ for matrices over $\mathbb{O}$.

Lemma 4.3. *Let $A \in \mathbb{O}^{n\times 1}$, $B \in \mathbb{O}^{1\times n}$ and $x \in \mathbb{O}$ be given. Then*

$$\mathrm{vec}\,(Ax) = \omega(A)\overrightarrow{x} \quad \textit{and} \quad \mathrm{vec}\,(xB) = \nu(B^T)\overrightarrow{x}. \tag{4.11}$$

Proof. Let $A = [\, a_1, \ \cdots, \ a_n \,]^T$ and $B = [\, b_1, \ \cdots, \ b_n \,]^T$. Then by Eqs.(2.3), (2.7) and Eqs.(4.1)—(4.3) we find

$$\mathrm{vec}(Ax) = \begin{bmatrix} \overrightarrow{a_1x} \\ \vdots \\ \overrightarrow{a_nx} \end{bmatrix} = \begin{bmatrix} \omega(a_1)\overrightarrow{x} \\ \vdots \\ \omega(a_n)\overrightarrow{x} \end{bmatrix} = \begin{bmatrix} \omega(a_1) \\ \vdots \\ \omega(a_n) \end{bmatrix} \overrightarrow{x} = \omega(A)\overrightarrow{x},$$

and

$$\mathrm{vec}(xB) = \begin{bmatrix} \overrightarrow{xb_1} \\ \vdots \\ \overrightarrow{xb_n} \end{bmatrix} = \begin{bmatrix} \nu(b_1)\overrightarrow{x} \\ \vdots \\ \nu(b_n)\overrightarrow{x} \end{bmatrix} = \begin{bmatrix} \nu(b_1) \\ \vdots \\ \nu(b_n) \end{bmatrix} \overrightarrow{x} = \nu(B^T)\overrightarrow{x}. \qquad \square$$

Lemma 4.4. *Let $A \in \mathbb{O}^{m\times n}$, $X \in \mathbb{O}^{n\times 1}$ and $a \in \mathbb{O}$ be given. Then*

$$\mathrm{vec}(AX) = \omega(A)\mathrm{vec}X \quad \textit{and} \quad \mathrm{vec}(Xa) = [\, \nu(a)\widehat{\otimes} I_{8n} \,]\mathrm{vec}X = \nu(a)\widehat{\otimes}\mathrm{vec}X. \tag{4.12}$$

Proof. Let $A = [\, A_1, \ \cdots, \ A_n \,]$ and $X = [\, x_1, \ \cdots, \ x_n \,]^T$. Then by Eq.(4.11) we find

$$\begin{aligned} \mathrm{vec}(AX) &= \mathrm{vec}(A_1x_1 + \cdots + A_nx_n) \\ &= \mathrm{vec}(A_1x_1) + \cdots + \mathrm{vec}(A_nx_n) \\ &= \omega(A_1)\mathrm{vec}x_1 + \cdots + \omega(A_n)\mathrm{vec}x_n \\ &= [\, \omega(A_1), \ \cdots, \ \omega(A_n) \,] \begin{bmatrix} \mathrm{vec}\, x_1 \\ \vdots \\ \mathrm{vec}\, x_n \end{bmatrix} = \omega(A)\mathrm{vec}X, \end{aligned}$$

as required for the first equality in (4.12). On the other hand,

$$\mathrm{vec}(Xa) = \begin{bmatrix} \overrightarrow{x_1a} \\ \vdots \\ \overrightarrow{x_na} \end{bmatrix} = \begin{bmatrix} \nu(a)\overrightarrow{x_1} \\ \vdots \\ \nu(a)\overrightarrow{x_n} \end{bmatrix} = [\, \nu(a)\widehat{\otimes} I_{8n} \,]\mathrm{vec}X = \nu(a)\widehat{\otimes}\mathrm{vec}X,$$

as required for the second equality in (4.12). $\square$

Lemma 4.5. *Let $B \in \mathbb{O}^{p\times 1}$ and $X \in \mathbb{O}^{n\times p}$ be given. Then*

$$\mathrm{vec}\,(XB) = [\, \nu(B^T)\widehat{\otimes} I_{8n} \,]\mathrm{vec}X. \tag{4.13}$$

Proof. Let $X = [\, X_1, \ \cdots, \ X_p\,]$ and $B = [\, b_1, \ \cdots, \ b_p\,]^T$. Then it follows from the second equality in (4.12) that

$$\begin{aligned} \mathrm{vec}(XB) &= \mathrm{vec}(X_1 b_1 + \cdots + X_p b_p) \\ &= \mathrm{vec}(X_1 b_1) + \cdots + \mathrm{vec}(X_p b_p) \\ &= (\nu(b_1)\widehat{\otimes} I_{8n})\mathrm{vec}X_1 + \cdots + (\nu(b_p)\widehat{\otimes} I_{8n})\mathrm{vec}X_p \\ &= (\,[\,\nu(b_1), \ \cdots, \ \nu(b_p)\,]\widehat{\otimes} I_{8n}\,) \begin{bmatrix} \mathrm{vec}X_1 \\ \vdots \\ \mathrm{vec}X_p \end{bmatrix} = [\nu(B^T)\widehat{\otimes} I_{8n}]\mathrm{vec}X, \end{aligned}$$

as required for Eq.(4.13). □

Based on the above several lemmas, we can find the following three general results.

Theorem 4.6. *Let $A = (a_{st}) \in \mathbb{O}^{m\times n}$ and $X \in \mathbb{O}^{n\times p}$ be given. Then*

$$\mathrm{vec}(AX) = [\, I_{8p}\widehat{\otimes}\omega(A)\,]\mathrm{vec}X. \tag{4.14}$$

Proof. Let $X = [\, X_1, \ \cdots, \ X_p\,]$. Then we find by Eq.(4.12) that

$$\begin{aligned} \mathrm{vec}(AX) &= \mathrm{vec}[\, AX_1, \ \cdots, \ AX_p\,] \\ &= [\,\mathrm{vec}(AX_1), \ \cdots, \ \mathrm{vec}(AX_p)\,] \\ &= [\,\omega(A)\mathrm{vec}X_1, \ \cdots, \ \omega(A)\mathrm{vec}X_p\,] \\ &= \mathrm{diag}(\,\omega(A), \ \cdots, \ \omega(A)\,)[\,\mathrm{vec}X_1, \ \cdots, \ \mathrm{vec}X_p\,] = [\, I_{8p}\widehat{\otimes}\omega(A)\,]\mathrm{vec}X, \end{aligned}$$

establishing Eq.(4.14). □

Theorem 4.7. *Let $B = (b_{st}) \in \mathbb{O}^{p\times q}$ and $X \in \mathbb{O}^{n\times p}$ be given. Then*

$$\mathrm{vec}(XB) = [\,\nu(B^T)\widehat{\otimes} I_{8n}\,]\mathrm{vec}X. \tag{4.15}$$

Proof. Let $B = [\, B_1, \ \cdots, \ B_q\,]$. Then we find by Eq.(4.13) that

$$\begin{aligned} &\mathrm{vec}(XB) = \mathrm{vec}[\, XB_1, \ \cdots, \ XB_q\,] \\ &= \begin{bmatrix} \mathrm{vec}XB_1 \\ \vdots \\ \mathrm{vec}XB_q \end{bmatrix} \\ &= \begin{bmatrix} [\,\nu(B_1^T)\widehat{\otimes} I_{8n}\,]\,\mathrm{vec}X \\ \vdots \\ [\,\nu(B_q^T)\widehat{\otimes} I_{8n}\,]\,\mathrm{vec}X \end{bmatrix} = \begin{bmatrix} [\,\nu(B_1^T)\widehat{\otimes} I_{8n}\,] \\ \vdots \\ [\,\nu(B_q^T)\widehat{\otimes} I_{8n}\,] \end{bmatrix} \mathrm{vec}X = [\,\nu(B^T)\widehat{\otimes} I_{8n}\,]\mathrm{vec}X, \end{aligned}$$

as required for Eq.(4.15). $\square$

Theorem 4.8. *Let $A = (a_{st}) \in \mathbb{O}^{m\times n}$, $B = (b_{st}) \in \mathbb{O}^{p\times q}$, and $X \in \mathbb{O}^{n\times p}$ be given. Then*

$$\mathrm{vec}[(AX)B] = [\,\nu(B^T)\widehat{\otimes}\omega(A)\,]\mathrm{vec}X, \quad \textit{and} \quad \mathrm{vec}[A(XB)] = [\,\omega(A)\tilde{\otimes}\nu(B^T)\,]\mathrm{vec}X. \tag{4.16}$$

Proof. According to Eqs.(4.14) and (4.15), we find that

$$\begin{aligned}\mathrm{vec}[(AX)B] &= [\,\nu(B^T)\widehat{\otimes}I_{8m}\,]\mathrm{vec}(AX)\\ &= [\,\nu(B^T)\widehat{\otimes}I_{8m}\,][\,I_{8p}\widehat{\otimes}\omega(A)\,]\mathrm{vec}X = [\,\nu(B^T)\widehat{\otimes}\omega(A)\,]\mathrm{vec}X,\end{aligned}$$

and

$$\begin{aligned}\mathrm{vec}[A(XB)] &= [\,I_{8p}\widehat{\otimes}\omega(A)\,]\mathrm{vec}(XB)\\ &= [\,I_{8p}\widehat{\otimes}\omega(A)\,][\,\nu(B^T)\widehat{\otimes}I_{8n}\,]\mathrm{vec}X = [\,\omega(A)\tilde{\otimes}\nu(B^T)\,]\mathrm{vec}X,\end{aligned}$$

as required for Eq.(4.16). $\square$

Theorem 4.9. *Let $A = (a_{st}) \in \mathbb{O}^{n\times n}$, $X = (b_{st}) \in \mathbb{O}^{n\times p}$, $Y \in \mathbb{O}^{q\times n}$ be given, and denote*

$$A^{(k|} * X = A(A\cdots(AX)\cdots)), \quad \textit{and} \quad Y * A^{|k)} = ((\cdots(YA)\cdots)A)A.$$

Then

$$\mathrm{vec}(A^{(k|} * X) = [\,I_{8p}\widehat{\otimes}\omega^k(A)\,]\mathrm{vec}X, \quad \textit{and} \quad \mathrm{vec}(Y * A^{|k)}) = [\,\nu^k(A^T)\widehat{\otimes}I_{8q}\,]\mathrm{vec}Y. \tag{4.17}$$

Just as the standard Kronecker products for matrices over any field, the three formulas in Eqs.(4.14)—(4.16) can directly be used for transforming any linear matrix equations over $\mathbb{O}$ into an ordinary linear system of equation over $\mathbb{R}$. For example,

$$\begin{aligned}
&AX = B \iff [\,I\widehat{\otimes}\omega(A)\,]\mathrm{vec}X = \mathrm{vec}B,\\
&XA = B \iff [\,\nu(A^T)\widehat{\otimes}I\,]\mathrm{vec}X = \mathrm{vec}B,\\
&A(BX) = C \iff [\,I\widehat{\otimes}\omega(A)\omega(B)\,]\mathrm{vec}X = \mathrm{vec}C,\\
&(XA)B = C \iff [\,\nu(B^T)\nu(A^T)\widehat{\otimes}I\,]\mathrm{vec}X = \mathrm{vec}C,\\
&(AX)B = C \iff [\,\nu(B^T)\widehat{\otimes}\omega(A)\,]\mathrm{vec}X = \mathrm{vec}C,\\
&A(XB) = C \iff [\,\omega(A)\tilde{\otimes}\nu(B)\,]\mathrm{vec}X = \mathrm{vec}C,\\
&AX - XB = C \iff [\,I\widehat{\otimes}\omega(A) - \nu(B)\widehat{\otimes}I\,]\mathrm{vec}X = \mathrm{vec}C,\\
&(AX)A - A(XA) = B \iff [\,\nu(A^T)\widehat{\otimes}\omega(A) - \omega(A)\tilde{\otimes}\nu(A^T)\,]\mathrm{vec}X = \mathrm{vec}B.
\end{aligned}$$

Theoretically speaking, various problems related to linear matrix equations over the octonion algebra now have a complete resolution.

Below are several simple results related to solutions of linear matrix equations over $\mathbb{O}$.

Definition 4.3. Let $A \in \mathbb{O}^{n\times n}$ be given. If its left adjoint matrix $\omega(A)$ is invertible, then A is said to be *completely invertible.*

Theorem 4.10. *Let $A = (a_{st}) \in \mathbb{O}^{m\times m}$ and $B = (b_{st}) \in \mathbb{O}^{m\times n}$ be given. If A is completely invertible, then the matrix equation*

$$AX = B, \tag{4.18}$$

has a unique solution over $\mathbb{O}$. In that case, if the real characteristic polynomial of $\omega(A)$ is

$$p(\lambda) = \lambda^t + r_{t-1}\lambda^{t-1} + \cdots + r_1\lambda + r_0, \tag{4.19}$$

where r_0 is the determinant of $\omega(A)$, then the unique solution of Eq.(4.18) *can be expressed as*

$$X = -\frac{1}{r_0}[\, A^{(t-1|} * B + r_{t-1}(A^{(t-2|} * B) + \cdots + r_3 A(AB) + r_2 AB + r_1 B\,]. \tag{4.20}$$

Proof. According to Eq.(4.14), the matrix equation (4.18) is equivalent to

$$[\, I_{8n}\widehat{\otimes}\omega(A)\,]\mathrm{vec}X = \mathrm{vec}B. \tag{4.21}$$

Because $\omega(A)$ is invertible, $I_{8m}\hat{\otimes}\omega(A)$ is also invertible. Hence the solution of Eq.(4.25) is unique and this solution is

$$\mathrm{vec}X = [\, I_{8n}\widehat{\otimes}\omega(A)\,]^{-1}\mathrm{vec}B = [\, I_{8m}\widehat{\otimes}\omega^{-1}(A)\,]\mathrm{vec}B.$$

Observe that

$$\omega^t(A) + r_{t-1}\omega^{t-1}(A) + \cdots + r_1\omega(A) + r_0 I_{8m} = 0$$

holds. We then have

$$\omega^{-1}(A) = -\frac{1}{r_0}\left[\, \omega^{t-1}(A) + r_{t-1}\omega^{t-2}(A) + \cdots + r_2\omega(A) + r_1 I_{8m}\,\right].$$

Thus

$$\begin{aligned} & I_{8n}\widehat{\otimes}\omega^{-1}(A) = \\ = & -\frac{1}{r_0}[\, I_{8n}\widehat{\otimes}\omega^{t-1}(A) + r_{t-1}(\, I_{8n}\widehat{\otimes}\omega^{t-2}(A)\,) + \cdots + r_2(\, I_{8n}\widehat{\otimes}\omega(A)\,) + \\ & +(\, r_1 I_{8n}\widehat{\otimes} I_{8m}\,)\,], \end{aligned}$$

and

$$\begin{aligned}\operatorname{vec} X &= [\, I_{8n}\widehat{\otimes}\omega(A)\,]^{-1}\operatorname{vec} B = \\ &= -\frac{1}{r_0}[\,(\, I_{8n}\widehat{\otimes}\omega^{t-1}(A)\,)\operatorname{vec} B + r_{t-1}(\, I_{8n}\widehat{\otimes}\omega^{t-2}(A)\,)\mathrm{vec}B \\ &\quad + \cdots + r_2(\, I_{8n}\widehat{\otimes}\omega(A)\,)\operatorname{vec} B + r_1(\, I_{8n}\widehat{\otimes}I_{8m}\,)\operatorname{vec} B\,].\end{aligned}$$

Returning it to octonion matrix expression by Eq.(4.17), we obtain Eq.(4.24). □
Similarly we have the following.

Theorem 4.11. *Let $A = (a_{st}) \in \mathbb{O}^{m\times m}$ and $B = (b_{st}) \in \mathbb{O}^{n\times m}$ be given. If A is completely invertible, then the matrix equation $XA = B$ has a unique solution over $\mathbb{O}$. In that case, if the real characteristic polynomial of $\omega(A)$ is*

$$p(\lambda) = \lambda^t + r_{t-1}\lambda^{t-1} + \cdots + r_1\lambda + r_0, \tag{4.22}$$

then the unique solution of $XB = A$ can be expressed as

$$X = -\frac{1}{r_0}[\, B*A^{|t-1)} + r_{t-1}(B*A^{|t-2)}) + \cdots + r_3(BA)A + r_2BA + r_1B\,]. \tag{4.23}$$

For simplicity, the two solutions in Eqs.(4.20) and (4.23) can also be written as

$$X = L_A^{-1}\circ B, \qquad X = B\circ R_A^{-1}, \tag{4.24}$$

where L_A^{-1} and R_A^{-1} are, respectively, called the *left and the right inverse operators* of the completely invertible octonion matrix A. Some properties on these two inverse operators are listed below.

Theorem 4.12. *Let $A \in \mathbb{O}^{m\times m}$ be a completely invertible matrix, $B \in \mathbb{O}^{m\times n}$ and $C \in \mathbb{O}^{n\times m}$ be given. Then*

$$A(L_A^{-1}\circ B) = B, \qquad A(L_A^{-1}\circ I_m) = I_m, \tag{4.25}$$

$$L_A^{-1}\circ(AB) = B, \qquad L_A^{-1}\circ A = I_m, \tag{4.26}$$

$$(C\circ R_A^{-1})A = C, \qquad (I_m\circ R_A^{-1})A = I_m, \tag{4.27}$$

$$(CA)\circ R_A^{-1} = C, \qquad A\circ R_A^{-1} = I_m. \tag{4.28}$$

Proof. Follows from Theorems 4.10 and 4.11. □

We can also consider the inverses of octonion matrices in the usual sense. Let $A \in \mathbb{O}^{m\times m}$ be given. If there are $X,\, Y \in \mathbb{O}^{m\times m}$ such that $XA = I_m$ and

$AY = I_m$, then X and Y are, respectively, called the *left inverse* and the *right inverse* of A, and denoted by $A_L^{-1} := X$ and $A_R^{-1} := Y$. From Theorems 4.10 and 4.11, we know that a square matrix of order m over $\mathbb{O}$ has a left inverse if and only if the equation $[\,\nu(A^T)\widehat{\otimes} I_{8m}\,]\text{vec}X = \text{vec}I_m$ is solvable, and A has a right inverse if and only if the equation $[\,I_{8m}\widehat{\otimes}\omega(A)\,]\text{vec}Y = \text{vec}I_m$ is solvable. These two facts imply that the left and the right inverses of a square matrix may not be unique, even both of them exist. As two special cases, we have the following.

Theorem 4.13. *Let $A \in \mathbb{O}^{m\times m}$ be given. Then the left and the right inverses of A are unique if and only if A is completely invertible. In that case, if the real characteristic polynomial of $\omega(A)$ is*

$$p(\lambda) = \lambda^t + r_{t-1}\lambda^{t-1} + \cdots + r_1\lambda + r_0,$$

then the unique left and the unique right inverses A can be expressed as

$$A_L^{-1} = -\frac{1}{r_0}[\,A^{(t-1|} + r_{t-1}A^{(t-2|} + \cdots + r_3A(A^2) + r_2A^2 + r_1I_m\,],$$

and

$$A_R^{-1} = -\frac{1}{r_0}[\,A^{|t-1)} + r_{t-1}A^{|t-2)} + \cdots + r_3(A^2)A + r_2A^2 + r_1I_m\,],$$

where $A^{(s|} := A(A(\cdots(AA)\cdots))$ and $A^{|s)} := ((\cdots(AA)\cdots)A)A$.

Proof. Follows directly from Theorems 4.10 and 4.11. $\square$

Based on Theorems 4.10 and 4.12, as well as Eqs.(4.25)—(4.28), we can also derive the following two simple results.

Corollary 4.14. *If $A \in \mathbb{O}^{m\times m}$ is completely invertible, and $AB_1 = AC_1$ and $B_2A = C_2A$, then $B_1 = C_1$ and $B_2 = C_2$. In other words, the left and the right cancellation rules hold for completely invertible matrices.*

Corollary 4.15. *Suppose that $A \in \mathbb{O}^{m\times m}$, $B \in \mathbb{O}^{n\times n}$ are completely invertible and $C \in \mathbb{O}^{m\times n}$. Then*

(a) *The matrix equation $A(XB) = C$ has a unique solution* $X = (L_A^{-1} \circ C)R_B^{-1}$.

(b) *The matrix equation $(AX)B = C$ has a unique solution* $X = L_A^{-1}(C \circ R_B^{-1})$, *where L_A^{-1} and R_B^{-1} are the left and the right inverse operators of A and B respectively.*

Our next result is concerned with the extension of the Cayley-Hamilton theorem to octonion matrices, which could be regarded as one of the most successful applications of matrix representations of octonions.

Theorem 4.16. *Let $A \in \mathbb{O}^{m\times m}$ be given and suppose that the real characteristic polynomial of $\omega(A)$ is*

$$p(\lambda) = \lambda^t + r_{t-1}\lambda^{t-1} + \cdots + r_1\lambda + r_0.$$

Then A satisfies the following two identities

$$A^{(t|} + r_{t-1}A^{(t-1|} + \cdots + r_3A(AA) + r_2A^2 + r_1A + r_0I_m = 0, \tag{4.29}$$

$$A^{|t)} + r_{t-1}A^{|t-1)} + \cdots + r_3(AA)A + r_2A^2 + r_1A + r_0I_m = 0. \tag{4.30}$$

Proof. Observe that $p[\omega(A)] = 0$. It follows that

$$[\, I_{8m}\widehat{\otimes}p[\omega(A)]\,]\mathrm{vec}I_m = 0. \tag{4.31}$$

On the other hand, it is east to see by Eq.(4.17) that

$$\mathrm{vec}A^{(s|} = \mathrm{vec}(A^{(s|} * I_m) = [\, I_{8m}\widehat{\otimes}\omega^s(A)\,]\mathrm{vec}I_m, \quad s = 1,\, 2,\, \cdots.$$

Thus we find that

$$\begin{aligned}
&[\, I_{8m}\widehat{\otimes}p(\omega(A))\,]\mathrm{vec}\, I_m \\
&= [\, I_{8m}\widehat{\otimes}\omega^t(A) + r_{t-1}(\, I_{8m}\widehat{\otimes}\omega^{t-1}(A)\,) + \cdots + \\
&\quad +r_1(\, I_{8m}\widehat{\otimes}\omega(A)\,) + \ r_0(\, I_{8m}\widehat{\otimes}I_{8m}\,)\,]\mathrm{vec}\, I_m \\
&= (\, I_{8m}\widehat{\otimes}\omega^t(A)\,)\mathrm{vec}I_m + r_{t-1}(\, I_{8m}\widehat{\otimes}\omega^{t-1}(A)\,)\mathrm{vec}\, I_m + \cdots + \\
&\quad +r_1(\, I_{8m}\widehat{\otimes}\omega(A)\,)\mathrm{vec}\, I_m + r_0(\, I_{8m}\widehat{\otimes}I_{8m}\,)\mathrm{vec}\, I_m \\
&= \mathrm{vec}\, A^{(t|} + r_{t-1}\mathrm{vec}\, A^{(t-1|} + \cdots + r_1\mathrm{vec}\, A + r_0\mathrm{vec}\, I_m \\
&= \mathrm{vec}[\, A^{(t|} + r_{t-1}A^{(t-1|} + \cdots + r_1A + r_0I_m\,].
\end{aligned}$$

The combination of this equality with Eq.(4.31) results in Eq.(4.29). The identity in Eq.(3.30) can be established similarly. □

Finally we present a result on real eigenvalues of Hermitian octonion matrices.

Theorem 4.17. *Suppose that $A \in \mathbb{O}^{m\times m}$ is Hermitian, that is, $A^* = A$. Then A and its real adjoint $\omega(A)$ have identical real eigenvalues.*

Proof. Since $A = A^*$, we know by Theorem 4.1(e) that $\omega(A) = \omega(A^*) = \omega^T(A)$, that is, $\omega(A)$ is a real symmetric matrix. In that case, all eigenvalues of $\omega(A)$ are real. Now suppose that

$$\omega(A)X = X\lambda, \tag{4.32}$$

where $\lambda \in \mathbb{R}$ and $X \in \mathbb{R}^{8m\times 1}$. Then there is unique $Y \in \mathbb{O}^{m\times 1}$ such that $\mathrm{vec}\, Y = X$. In that case, it is easy to find by Theorem 4.1(a) and Eq.(4.12) that

$$\omega(A)X = X\lambda \Longrightarrow \omega(A)\mathrm{vec}\, Y = \mathrm{vec}\, Y\lambda \Longrightarrow \mathrm{vec}\,(AY) = \mathrm{vec}\,(Y\lambda) \Longrightarrow AY = Y\lambda, \tag{4.33}$$

which implies that λ is a real eigenvalue of A, and Y is an eigenvector of A corresponding to this λ. Conversely suppose that $AY = Y\lambda$, where $\lambda \in \mathbb{R}$, $Y \in \mathbb{O}^{m\times 1}$. Then taking vec operation on its both sides according to Eq.(4.12) yields

$$\omega(A)\mathrm{vec}\, Y = \mathrm{vec}\, Y\lambda.$$

This implies that λ is also a real eigenvalue of $\omega(A)$ and $\mathrm{vec}\, Y$ is a real eigenvector of $\omega(A)$ associcated with this λ. □

The above result clearly shows that real eigenvalues and the corresponding eigenvectors of a Hermitian octonion matrix A can all be determined by its real adjoint $\omega(A)$. Since $\omega(A)$ is a real symmetric $8m \times 8m$ matrix, it has $8m$ eigenvalues and $8m$ corresponding orthogonal eigenvectors.

Now a fundamental problem would naturally be asked: how many different real eigenvalues can a Hermitian octonion matrix A have at most? For a 2×2 Hermitian octonion matrix $A = \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix}$, where $a,\ c \in \mathbb{R}$, its real adjoint is

$$\omega(A) = \begin{bmatrix} aI_8 & \omega(b) \\ \omega^T(b) & cI_8 \end{bmatrix}.$$

Clearly the characteristic polynomial of $\omega(A)$ is

$$\det(\, \lambda I_{16} - \omega(A)\,) = [\,(\lambda - a)(\lambda - c) - |b|^2\,]^8.$$

This shows that $\omega(A)$, and correspondingly A, has 2 eigenvalues, each of which has a multiplicity 8.

The eigenvalue problem for 3×3 Hermitian octonion matrices was recently examined by Dray and Manogue [6] and Okubo [11]. They showed by algebraic methods that every 3×3 Hermitian octonion matrix has 24 real eigenvalues which are divided into 6 groups, each of them has multiplicity 4. Now according to Theorem 4.17, the real eigenvalues of any 3×3 Hermitian octonion matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \overline{a}_{12} & a_{22} & a_{23} \\ \overline{a}_{13} & \overline{a}_{23} & a_{33} \end{bmatrix}, \qquad a_{11},\ a_{22},\ a_{33} \in \mathbb{R},$$

can be completely determined by its real adjoint

$$\omega(A) = \begin{bmatrix} \omega(a_{11}) & \omega(a_{12}) & \omega(a_{13}) \\ \omega^T(a_{12}) & \omega(a_{22}) & \omega(a_{23}) \\ \omega^T(a_{13}) & \omega^T(a_{23}) & \omega(a_{33}) \end{bmatrix}.$$

Obviously this matrix has 24 real eigenvalues and the 24 corresponding real orthogonal eigenvectors. Numerical computation shows that these 24 eigenvalues are divided into 6 groups, each of them has multiplicity 4, which is consistent with the fact revealed in [6] and [11]. Moreover the 24 real orthogonal eigenvectors can also be converted to octonion expressions by (4.33).

Furthermore, numerical computation reveals an interesting fact that the 32 real eigenvalues any 4×4 Hermitian octonion matrix are divided into 16 groups, each of them has multiplicity 2; the 40 real eigenvalues of any 5×5 Hermitian octonion matrix are divided into 20 groups, each of them has multiplicity 2.

In general, we guess that for any $m \times m$ Hermitian octonion matrix with $m > 3$, its $8m$ real eigenvalues can be divided into $4m$ groups, each of them has multiplicity 2.

As a subsequent work of Thereom 4.17, one might naturally ask how to establish a possible factorization for a Hermitian octonion matrix using its real eigenvalues and corresponding octonion eigenvectors, speak more precisely, for an $m \times m$ Hermitian octonion matrix A, how to construct a complete invertible octonion matrix P (unitary?) and a real diagonal matrix D such that $A = PDP^{-1}$ using its $8m$ real eigenvalues and $8m$ corresponding octonion eigenvectors. For the 3×3 case, the problem was completely solved by Dray and Manogue in [6]. But we can say nothing at present for $m > 3$.

As pointed out by Dray and Manogue in [6], Hermitian octonion matrices can also have non-real right eigenvalues. Theoretically speaking, the non-real eigenvalue problem of Hermitian octonion matrices may also be converted to a problem related to real representations of octonion matrices. In fact, suppose that $AX = X\lambda$, where $\lambda \in \mathbb{O}$ and $X \in \mathbb{O}^{m\times 1}$. Then according to Eq.(4.12), it is equivalent to

$$\omega(A)\mathrm{vec}\, X = \nu(\lambda)\widehat{\otimes}\mathrm{vec}\, X,$$

or alternatively

$$[\,\omega(A) - \mathrm{diag}(\,\nu(\lambda),\ \cdots,\ \nu(\lambda)\,]\mathrm{vec}\, X = 0.$$

How to find $\nu(\lambda)$ satisfying the equation remains to further study.

Conclusions. In this paper, we have introduced two pseudo real matrix representations for octonions. Based on them we have made a complete investigation to their operation properties and have considered their various applications to octonions and matrices of octonions. However our work could only be regarded as a first step in the research of octonion matrix analysis and its applications. Numerous problems related to matrices of octonions remain to further examine, such as:

(a) How to determine eigenvalues and eigenvectors of a square octonion matrix, not necessarily Hermitian, and what is the relationship of eigenvalues and eigenvectors of an octonion matrix and its real adjoint matrices?
(b) Besides Eq.(4.29) and (4.30), how to establish some other identities for octonion matrices through their adjoint matrices?
(c) How to establish similarity theory for octonion matrices, and how to determine the relationship between the similarity of octonions matrices and the similarity of their adjoint matrices?
(d) How to consider various possible decompositions of octonion matrices, such as, LU decomposition, singular value decomposition and Schur decomposition?
(e) How to characterize various particular octonion matrices, such as, idempotent matrices, nipoltent matrices, involutary matrices, unitary matrices, normal matrices, and so on?
(f) How to define generalized inverses of octonion matrices when they are not completely invertible?

and so on. As mentioned in the beginning of the section, matrix multiplication for octonion matrices is completely not associative. In that case, any further research to problems related matrices of octonions is extremely difficult, but is also quite challenging. Any advance in solving the problems mentioned above could lead to remarkable new development in the real octonion algebra and its applications in mathematical physics.

Finally we should point out that the results obtained in the paper can be used to establish pseudo matrix representations for real sedenions, as well as, in general, for elements in any 2^n-dimensional real Cayley-Dickson algebras.

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