Chapter 3 Fractions and Decimals

3.1 Fractions

Fractions maybe defined as part of a whole. Over 5 000 years ago, the Egyptians defined fractions as:

$\frac{1}{2}$ is that, 2 of which makes a whole.

$\frac{1}{3}$ is that, 3 of which makes a whole.

$\frac{1}{4}$ is that, 4 of which makes a whole.

And for any counting or natural number denoted by n :

$\frac{1}{n}$ is that, n of which makes a whole.

Fractions of the above form are unit fractions, and a general fraction is the number of unit fractions.

Example $\frac{5}{7}$ is 5 of the unit fractions $\frac{1}{7}$. That is 5, of that which 7 makes a whole.

So, in general $\frac{\mathrm{m}}{\mathrm{n}}$ is m of the unit fractions $\frac{1}{\mathrm{n}}$. That is m, of that which n makes a whole.

Comments: The whole is the unit and depending on the context, it could be one, two or more items. A unit fraction of the whole is one part of the whole divided equally into the number of the unit fractions, which makes the whole.

Example, $\frac{1}{9}$ of a whole, is the whole divided into 9 equal parts and one of these parts.

$\frac{1}{n}$ of a whole, is the whole divided into n equal parts, and one of these parts.

Fractions are also called rational numbers. Rational numbers are constructed with integers. They are numbers of the form, $\frac{m}{n}$, where both "m" and "n" are integers, but "n" cannot be zero.

Examples: $\frac{1}{2}, \frac{306}{217}, \frac{5}{1}, \frac{^{-}12}{13}, \frac{0}{23}, \frac{78}{78}$

Irrational Numbers: This arose from questions inspired by the multiplication of a natural number by itself or square of a number.

$$1 \times 1 = 1$$
$$2 \times 2 = 4$$
$$3 \times 3 = 9$$
$$4 \times 4 = 16$$
$$\vdots$$

Given a number we find its square by multiplying the number by itself. This led to the question of finding the number whose square will be equal to a given number. That is problems of the form:

$$\square \times \square = 25$$
$$\square \times \square = 49$$
$$\square \times \square = 121$$

The solution could be found by substituting various natural numbers until the correct one is obtained, if the right side is the square of a natural number. What happens if the right side is not the square of a natural number? For example, it is clear that the solution of, $\square \times \square = 2$ cannot be a natural number. That is because 2 is not the square of a natural number. It was, however, wrongly assumed that the solution was a rational number. Since the solution is neither an integer nor a rational number, it was given the name Irrational Number. It turned out that there are more irrational numbers than all the other numbers combined. In practice, irrational numbers are approximated by rational numbers.

3.2 Arithmetic of Fractions

$Re\min der$: Fractions are numbers of form: $\frac{N}{D}$, N is any Integer, D is any integer other than zero.

The line between N and D is called the fraction line. The Numerator is the number above the line – N. The Denominator is the number below the line – D. Fractions with the same denominator are of the same type. In practice, the Numerator and Denominator has 1–one, as the only common factor.

3.3 Multiplication of Fractions: The product of two fractions is the fraction with Numerator, the product of the numerators, and Denominator, the product of the denominators of the two fractions.

Example: $\frac{3}{4} \times \frac{5}{7} = \frac{3\times 5}{4\times 7} = \frac{15}{28}$; $\quad \frac{{}^{-}11}{13} \times \frac{2}{9} = \frac{{}^{-}11\times 2}{13\times 9} = \frac{{}^{-}22}{117}$

For the fractions $\frac{n}{m}$ *and* $\frac{p}{q}$: $\quad \frac{n}{m} \times \frac{p}{q} = \frac{n\times p}{m\times q}$

3.4 Division of Fractions Division is expressed in terms of multiplication, by the reciprocal of the divisor, and then the rule for multiplication is applied to complete the division.

Example: $\frac{3}{4} \div \frac{5}{7} = \frac{3}{4} \times \frac{7}{5}$, $\quad \frac{3}{4} \times \frac{7}{5} = \frac{21}{20} \quad \therefore \frac{3}{4} \div \frac{5}{7} = \frac{21}{20}$

$\frac{{}^{-}11}{13} \div \frac{9}{2} = \frac{{}^{-}11}{13} \times \frac{2}{9}$, $\quad \frac{{}^{-}11}{13} \times \frac{2}{9} = \frac{{}^{-}22}{117} \quad \therefore \frac{{}^{-}11}{13} \div \frac{9}{2} = \frac{{}^{-}22}{117}$

For the fractions $\frac{n}{m}$ *and* $\frac{p}{q}$: $\quad \frac{n}{m} \div \frac{p}{q} = \frac{n}{m} \times \frac{q}{p}$

3.5 Addition and Subtraction of Fractions

1. The sum of fractions with the same Denominator is the fraction is the sum of the numerators, over the Denominator of the fractions. The difference of fractions with the same Denominator is the difference between the numerators, over the Denominator of the fractions.

Example: $\frac{4}{7} + \frac{6}{7} = \frac{4\ +\ 6}{7}$, $\quad \therefore \quad \frac{4}{7} + \frac{6}{7} = \frac{10}{7}$

$\frac{29}{35} - \frac{17}{35} = \frac{29\ -\ 17}{35}$, $\quad \therefore \quad \frac{29}{35} - \frac{17}{35} = \frac{12}{35}$

For the fractions, $\frac{a}{d}$ *and* $\frac{c}{d}$: *Addition* $\frac{a}{d} + \frac{c}{d} = \frac{a+c}{d}$; $\quad$ *Subtraction* $\frac{a}{d} - \frac{c}{d} = \frac{a-c}{d}$

Addition and Subtraction of Fractions with different Denominators:

To get a single value for the sum of money in different denominations, the different denominations must be converted a common denomination. Example to get a single value by adding 100 cents to 1 dollar, if the cents is converted to dollars the sum is 2 dollars, or if the dollar is converted to cents the sum is 200 cents, or if the dollar and then the cents is converted to quarters the sum is eight quarters.

In the same way to get a single value by adding or subtracting fractions with different denominators, each of the fractions must be converted to an equivalent fraction with a common denominator. Once this is done, the rule for adding or subtracting fractions with the same denominator is applied to complete the addition or subtraction. The main task then is finding this common denominator. The method(s) of finding the common denominator is shown by examples.

Examples: *Find the sum*: $\frac{4}{7} + \frac{1}{5}$

$$\frac{4}{7} = \frac{4}{7} \times \frac{5}{5} = \frac{20}{35}, \quad \frac{1}{5} = \frac{1}{5} \times \frac{7}{7} = \frac{7}{35}$$

$$So: \frac{4}{7} + \frac{1}{5} = \frac{20}{35} + \frac{7}{35}, \quad \frac{20}{35} + \frac{7}{35} = \frac{20+7}{35}$$

$$\therefore \quad \frac{4}{7} + \frac{1}{5} = \frac{27}{35}$$

Find the difference $\frac{4}{7} - \frac{1}{5}$

$$\frac{4}{7} = \frac{4}{7} \times \frac{5}{5} = \frac{20}{35}, \quad \frac{1}{5} = \frac{1}{5} \times \frac{7}{7} = \frac{7}{35}$$

$$So: \frac{4}{7} - \frac{1}{5} = \frac{20}{35} - \frac{7}{35}, \quad \frac{20}{35} - \frac{7}{35} = \frac{20-7}{35}$$

$$\therefore \quad \frac{4}{7} - \frac{1}{5} = \frac{13}{35}$$

Find the sum: $\frac{9}{11} + \frac{7}{6}$

$$\frac{9}{11} = \frac{9}{11} \times \frac{6}{6} = \frac{54}{66}, \quad \frac{7}{6} = \frac{7}{6} \times \frac{11}{11} = \frac{77}{66}$$

$$So: \frac{9}{11} + \frac{7}{6} = \frac{54}{66} + \frac{77}{66}, \quad \frac{54}{66} + \frac{77}{66} = \frac{54+77}{66}$$

$$\therefore \quad \frac{9}{11} + \frac{7}{6} = \frac{131}{66}$$

Find the sum: $\frac{9}{11} - \frac{7}{6}$

$$\frac{9}{11} = \frac{9}{11} \times \frac{6}{6} = \frac{54}{66}, \quad \frac{7}{6} = \frac{7}{6} \times \frac{11}{11} = \frac{77}{66}$$

$$So: \frac{9}{11} - \frac{7}{6} = \frac{54}{66} - \frac{77}{66}, \quad \frac{54}{66} - \frac{77}{66} = \frac{54-77}{66}$$

$$\therefore \quad \frac{9}{11} - \frac{7}{6} = \frac{^{-}23}{66}$$

Procedure:

1. Multiply the first fraction by the fraction with both the numerator and the denominator, equal to the denominator of the second fraction.
2. Multiply the second fraction by the fraction with both the numerator and the denominator, equal to the denominator of the first fraction.
3. (The denominators of these 'new' fractions are equal.) The sum of the 'original' fractions is equal to the sum of these 'new' fractions, and the difference of the 'original' fractions is equal to the difference of these 'new' fractions.

For the sum: $\frac{a}{b} + \frac{p}{q}$: $\frac{a}{b} = \frac{a}{b} \times \frac{q}{q} = \frac{a \times q}{b \times q}$; $\frac{p}{q} = \frac{p}{q} \times \frac{b}{b} = \frac{p \times b}{q \times b}$

$b \times q = q \times b$ (multiplication is commutative) So, $\frac{a \times q}{b \times q} + \frac{p \times b}{q \times b} = \frac{a \times q + p \times b}{b \times q}$

$\therefore \quad \frac{a}{b} + \frac{p}{q} = \frac{a \times q + p \times b}{b \times q}$ (For subtraction change the addition sign to subtraction)

3.6 Fractions and Problem Solving

Some problems involve fractions, and we have to apply the procedures we have learned in adding, subtracting, multiplying, and dividing fractions to solve them.

Example 1

BJ Jewelry store sells diamonds for $1,200 per carat. The number of carats in each of the five diamonds sold are $\frac{1}{4}, \frac{5}{6}, \frac{3}{7}, \frac{1}{3}, \frac{1}{2}$. What is their total sale value?

Solution

First, we add up the fractions to find the total number of carats of the five diamonds. To add up fractions with different denominators, we must know a common denominator. We can find a common denominator by multiplying the different denominators together: 4 x 6 x 7 x 3 x 2 = 1008. And then divide each denominator into this common denominator to find the corresponding numerator.

$\frac{1}{4}+\frac{5}{6}+\frac{3}{7}+\frac{1}{3}+\frac{1}{2}=\frac{252}{1008}+\frac{840}{1008}+\frac{432}{1008}+\frac{336}{1008}+\frac{504}{1008}=\frac{2364}{1008}$ = 2.35 carats

Sale value of1 carat is $1,200

Therefore Sale value of 2.35 carats is 1,200 x 2.35 = $2,814.29

So, the total Sale value of the five diamonds is $2,814.29

Example 2

ABC Computer Services allocated its yearly profit as follows: $\frac{1}{6}$ or $1,200,000 reserve fund, $\frac{1}{4}$ or $300,000 as dividend, and the rest is to be ploughed backed into operation. What was the company's yearly profit?

Solution

We can proceed in two ways. First, we should note that out of a whole, $\frac{1}{6}+\frac{1}{4}$ were allocated to reserve and dividend. So, $1-(\frac{1}{6}+\frac{1}{4})=1-(\frac{2}{12}+\frac{3}{12})=1-\frac{5}{12}$.

We are left with $\frac{7}{12}$. If $\frac{5}{12}$ amounts to$1,500,000 (1,200,000 +$300,000), then $\frac{7}{12}$ amounts to what? Since $\frac{7}{12}$ is greater than $\frac{5}{12}$, we multiply $\frac{7}{12}$ by 1,5000 and divide by $\frac{5}{12}$. That is, $\frac{5}{12}$ carats sells for $1,500,000.

Here is the calculation for $\frac{7}{12}$: $1{,}500{,}000 \times \frac{7}{12} \div \frac{5}{12} = 1{,}500{,}000 \times \frac{7}{12} \times \frac{12}{5}$

$=1{,}500{,}000 \times \frac{7}{5}=\frac{10{,}500{,}000}{5}$ = $2,100,000

Note that, $\frac{5}{12} + \frac{7}{12} = \frac{12}{12} = 1$

$\Downarrow \quad \Downarrow = \Downarrow$

$ 1,500,000 + $2,100,000 = $3,600,000

Therefore, the yearly profit of the company is \$3,600,000 .

Alternatively, to find the company's yearly profit we can let the profit equal to 1 whole. This follows from the definition of fraction, which means part of a whole. Part ($\frac{1}{6}$) of the profit was allocated to reserve fund and another part ($\frac{1}{4}$) was given for distribution to shareholders. What is the whole? We proceed as follows:

$\frac{5}{12} = 1{,}500{,}000$

1 carat $= 1{,}500{,}000 \div \frac{5}{12} = 1{,}500{,}000 \text{ x } \frac{12}{5} = \frac{18{,}000{,}000}{5} = \$3{,}600{,}000$

Another alternative uses algebra. Let x represent the company's yearly profit. Then,

$\frac{5}{12}x = 1{,}500{,}000$ Multiply each side of the equation by $\frac{12}{1}$ to clear the fraction.

$\frac{5}{12}x \times \frac{12}{1} = 1{,}500{,}000 \times \frac{12}{1}$

$5x = 18{,}000{,}000$ Divide each side by 5

$x = \$3{,}600{,}000$

3.7 Review, Exercises, and Assignments

1. A tailor uses $3\frac{1}{2}$ m of material to make a weeding suit. A bolt contains 60m of material. How many suits can be made from the bolt of material?

2. In Joe's Pizza Hut a slice of pizza is $\frac{1}{12}$ of a whole pizza. Joe noted that there was $\frac{2}{3}$ of a whole pizza left. A customer ordered 9 slices of pizza. Could Joe serve the customer without baking a new pizza? Explain your answer.

3. Together Shirley and Mary weigh 190 kg. Shirley weighs $\frac{3}{4}$ as much as Mary. How much does each weigh?

4. A city hall's parking lot holds 64 cars. The parking lot is $\frac{7}{8}$ full. How many parking spaces remain in the lot?

5. Express each of the following decimals as fractions.
a) 0.085   b) 3.25   c) 12.74   d) 0.84   e) 2.75   f) 0.00025

6. Write each fraction in its lowest terms.
a) $\frac{12}{40}$ *b)* $\frac{10}{25}$ *c)* $\frac{14}{56}$ *d)* $\frac{48}{96}$ *e)* $\frac{120}{210}$

7. List these fractions in order of size, starting with the smallest.
$\frac{1}{3}, \frac{3}{9}, \frac{5}{9}, \frac{1}{6}$.

8. Dora receives $480 for part-time work that she did. She spends $280 on traveling to the US, and $80 on lunch. What fraction of her total wage was left?

9. Mrs. Johnson owned $\frac{3}{8}$ of a store. She sold $\frac{2}{3}$ of her interest in the store for $20 000. What was the total value of the store?

10. In Ocean community college $\frac{1}{3}$ of the students eat at the cafeteria, $\frac{1}{2}$ bring packed lunch and the rest eat their lunch outside the college. What fraction of students eat lunch outside the college?

11. Megan has filled $\frac{3}{5}$ of the space on her computer hard drive with data. She wants to keep $\frac{1}{4}$ of the hard drive free from data. What fraction of the hard drive is left free of data?

12.Simplify each of the following.
a) $\frac{1}{6} + \frac{3}{8}$ b) $\frac{5}{7} + \frac{2}{5}$ c) $\frac{11}{12} - \frac{3}{8}$ d) $\frac{3}{4} - \frac{2}{3}$
e) $\frac{1}{7} - \frac{1}{10}$ f) $\frac{5}{7} + \frac{5}{8}$

13. Simplify each of the following.

a) $\frac{3}{4} \times \frac{5}{7}$ b) $5\frac{1}{2} \times 1\frac{3}{4}$ c) $\frac{3}{4} \div \frac{1}{2}$

d) $\frac{3}{7} \div \frac{9}{10}$ e) $1\frac{2}{7} \times 1\frac{3}{8}$ f) $1\frac{1}{7} \div 2\frac{3}{8}$

14. It takes 3 hours to fill $\frac{2}{5}$ of a swimming pool. At this rate, how long will it take to fill the whole swimming pool?

15. What is the value of the product $(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4})$?

16. Pat paid $\frac{1}{3}$ of his debt one year and $\frac{1}{4}$ of his debt the second year.

a) At the end of the second year, how much fraction of his debt remained?

b) If the total balance of his debt was $7,000, how much did he pay?

c) How much was the original debt?

17. Mr. Lee spends a quarter of his net income on housing, one-fifth on food, one-tenth on clothing and one-fifth on other consumer expenses. He has $600 per month left for savings and retirement. What is his net income?

18. Four office desks that are $4\frac{1}{8}$ feet long are to be placed together on a wall that is $16\frac{5}{8}$ feet long. Will they fit on the wall? Explain your answer.

19. Which of the following pair of fractions are equivalent, and which are not equivalent?

a) $\frac{1}{3}, \frac{9}{18}$ b) $\frac{1}{5}, \frac{10}{20}$ c) $\frac{1}{2}, \frac{16}{32}$ d) $\frac{400}{600}, \frac{2}{3}$ e) $\frac{20}{100}, \frac{25}{50}$ f) $\frac{17}{34}, \frac{1}{2}$ g) $\frac{12}{64}, \frac{7}{42}$

20. What is the minimum number of identical squares tiles required completely to tile a rectangular living room with the dimensions $6\frac{1}{5}$ units by $3\frac{1}{5}$ units?

21.Ann worked the following hours in the last week of September: $7\frac{3}{4}, 6\frac{1}{2}, 7\frac{1}{4}, 9\frac{3}{4}$, and $8\frac{3}{4}$. Arlene worked 4.5 hours each day for six days. Who worked more hours? How many hours more?

22. Write each of the following fractions as decimal.

a) $\frac{5}{8}$ b) $\frac{3}{4}$ c) $\frac{4}{5}$ d) $\frac{2}{3}$ e) $\frac{3}{10}$ f) $\frac{1}{5}$ g) $\frac{6}{100}$ i) $\frac{3}{100}$

3.7 Answers to Fractions and Decimals- Review, Exercises and Assignments

1. 17 suits 2. 8 pieces 3. Mary weighs 104 kg; Shirley weighs 78kg.

4. 8 parking spaces. 5. a) 85/1000, b) 325/100, c) 1274/100, d) 84/100 e)275/100, f) 25/100000.

6. a) 3/10 b) 2/5 c) ¼ , d) ½ , e) 4/7

7. 1/6, 1/3, 4/9, 5/10 8. ¼. 9. $80000 10. 1/6, 11. 2/5

12. a) 13/24, b) 39/35, c) 13/24 d) 1/12 e) 3/70, f) 75/56

13. a) 5/7, b) 77/6, c) 3/2, d) 10/21, e) 99/56, f) 64/133.

14. $7\frac{1}{2}$ hours 15. ¼ 16. a) 5/12, b) $9800, c) $16800. 17. $2400 Change 18. Yes, $12\frac{1}{2}$ ft will be left 19. Equivalent: c, d, f; not equivalent:a, b, e, g.

20. 20 tiles 54. $7.33, 55. 150. 21. Ann worked 5 hours more.

22. a) 0.63, b) 0.75, c) 0.80, d) 0.67, e) 0.30, f) 0.20, g) 0.06, i) 0.03.