RELATIONS AND FUNCTIONS

Elements within a set or elements in different sets often have a special connection with one another that can be described as a relation

3.1. Binary relations

If we learn that two people, Bob and Pauline, are related, we understand that there is ~~samted~~ some family connection between them – that (Bob, Pauline) stands out from other ordered pairs of people because there is a relationship. (cousins, sister and brother, or whatever) The mathematical analog is to distinguish certain ordered pairs of objects from other ordered pairs because the first ones satisfy some relationship that the others do not.

3.1.1 Definition: Given sets S and T, a binary relation ρ (or $\mathcal{R}$) on $S \times T$ is a subset of $S \times T$

Exple: (i) $S = \{1, 2\}$ and $T = \{2, 3\}$

Then: $S \times T = \{(1,2), (1,3), (2,2), (2,3)\}$.

If we are interested in Equality $\{(2,2)\}$

" " " $<$ or $>$ $\{(1,2), (1,3), (2,3)\}$

(ii) $S = \{1, 2\}$, $T = \{2, 3, 4\}$

$\mathcal{R}$: $x \mathcal{R} y \Leftrightarrow x + y$ is odd

Then: $S \times T = \{(1,2), (1,3), (1,4), (2,2), (2,3), (2,4)\}$.

then $(1,2) \in \mathcal{R}$, $(1,4) \in \mathcal{R}$, and $(2,3) \in \mathcal{R}$

(iii) ρ on $\mathbb{N} \times \mathbb{N}$

(a) $x \rho y \Leftrightarrow x = y + 1$; $(2,2), (2,3), (3,3), (3,2)$

(b) $x \rho y \Leftrightarrow x$ divides y $(2,4), (2,5), (2,6)$

(iv) $|A| = m$, $|B| = n$

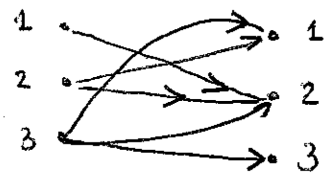
How many relations are there between A and B?

Relation could be any subset of Product set. $|A \times B| = m.n$
and $|\mathcal{P}[A \times B]| = 2^{m.n}$

3.1.2 Graphical representation of Relations:

We saw already that a relation can be represented in two ways, either in English, such as "greater than or equal", or by listing the set of ordered pairs belonging to the relation. A third way is to draw a graph in which arrows between symbols correspond to the ordered pairs. All three representation are equivalent.

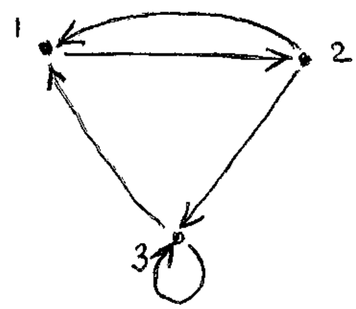
Exple,



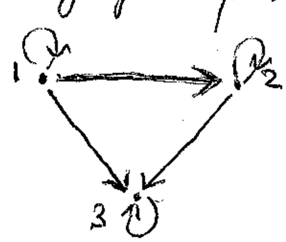
$R = \{(1,2), (2,1), (2,2)$
$(3,1); (3,2), (3,3)\}$.

if the relation is within a set A instead 2 copies of A, we we use the concept of Directed graph.

Exple: (i) $R = \{(1,2), (2,1), (2,3), (3,1), (3,3)\}$.



(ii) Draw the directed graph representation of the relation "$\leq$" in S_3;



3.1.3 Matrix Representation of Relations:

$S = \{s_1, s_2, \ldots, s_n\}$.
$T = \{t_1, t_2, \ldots, t_m\}$

Let $M(i,j) = \begin{cases} \text{false} & \text{if } (a_i, b_j) \notin R \\ \text{True} & \text{if } (a_i, b_j) \in R \end{cases}$

$M(i,j)$ logical matrix for R.

Exple: $S = \{1, 2, 3, 4\}$.
$T = \{a, b, c\}$.

$R = \{(1,b), (1,c), (2,a), (3,a), (3,b), (3,c), (4,b), (4,c)\}$.

$$M = \begin{array}{c|c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} F & T & T \\ T & F & F \\ T & T & T \\ F & T & T \end{bmatrix} \end{array}$$

3.2. Properties of Relations:

A binary relation on a set may have certain properties.

Let R be a relation on a set A:

R is Reflexive if for all $x \in A$, $x R x$

R is symmetric if for all $x, y \in A$, $x R y \Rightarrow y R x$

R is transitive if for all $x, y \in A$, $x R y$ and $y R z \Rightarrow x R z$

R is antisymmetric if for all $x, y \in A$, $x R y$ and $y R x \Rightarrow x = y$

Examples: Test the following binary Relations on the given sets S for reflexivity, Symmetry, antisymmetry and transitivity.

(a) $S = \mathbb{N}$; $x \rho y \Leftrightarrow x + y$ is even $R, S, \bar{A}, T$

(b) $S = \mathbb{N}$; $x \rho y \Leftrightarrow x$ divides y. $R, \bar{S}, A, T$

(c) $S = \{$all lines in the plane$\}$; $x \rho y \Leftrightarrow x$ is parallel to y or coincides with y.
$R, S, \bar{A}, T$

(c) ~~on~~ $S = \{1,2,3\}$, $\rho = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$.

$[1] = [2] = \{1,2\}$, $[3] = \{3\}$.

Theorem 3.1 : Let $S \neq \phi$, Let R be equ. relation on S. Then $\{[x] ; x \in S\}$ is a partition of S.

Conversely, given a particular π of S we can define the relation R_π on S.

$$R_\pi = \{(x,y) | ; [x] = [y]\}.$$

Then R_π is Equ. rel. on S

Exple : Let $S = \{1,2,3,4\}$.

$$\pi = \{\{1,2\},\{3\},\{4\}\}.$$

$$R_\pi = \{(1,2),(2,1),(1,1),(2,2),(3,3),(4,4)\}.$$

3.2.2. Order Relations - Partial ordering

A relation on a set S which is Reflexive, Antisymmetric and transitive is called Partial ordering on the set. The reason we have Partial is that not every pair of elements of A must be related.

Exple :

on $\mathbb{N}$; $x \rho y \Leftrightarrow x \leq y$.

on $\mathcal{P}(\mathbb{N})$: $A \rho B \Leftrightarrow A \subseteq B$

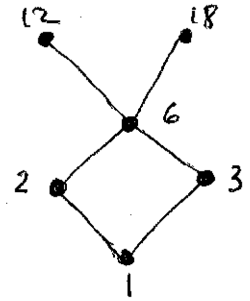
on $\{0,1\}$: $x \rho y \Leftrightarrow x = y^2$

If ρ is a partialordering on S then the ordered pair (S, ρ) is called patially ordered set or Poset

Exple : consider the relation "x divides y" on $\{1,2,3,6,12,18\}$. write the ordered pairs.

(1,1),(1,2),(2,2),(1,3),(3,3),(1,6),(6,6),(1,12),(12,12),(1,18)
(18,18),(2,6),(2,12),(2,18),(3,6),(3,12),(3,18),(6,12),(6,18)

graph. (Hasse Diagram) -



3-2.3 - <u>Composition of Relations</u>:

Read it if you want (not compulsory).

3-3 <u>Functions</u> : functions are assentially special cases of binary relations.

3.3.1 <u>Definition</u>: A function from a set A to set B is a relation f between A and B satisfying conditions:

(1) $\forall x \in A \; , \exists y \in B$ s.t $(x, y) \in f$

(2) $\forall x \in A \; , y, z \in B \; , (x,y) \in f$ and $(x,z) \in f$

$$\Rightarrow \quad y = z \, -$$

in other words a fct (mapping) $f: A \to B$ is a subset of $A \times B$ where each member of A appears exactly once.

A: Domain

B: Codomain.

If $(a,b) \in f \Rightarrow f(a) = b$ — b image of a, a preimage of b.

and f is said to map a to b.

<u>Examples</u>: (a) $f: S \to T$ where $S = T = \{1,2,3\}$.

$f = \{(1,1),(2,3),(3,1),(2,1)\}$.

Not a fct.

(b) $g: \mathbb{Z} \to \mathbb{N}$, $g(x) = |x|$ function

(c) S: set of all people in your Hometown - T is the set of SS#

$f: S \to T$ associates person to his SS#.

Yes and No.

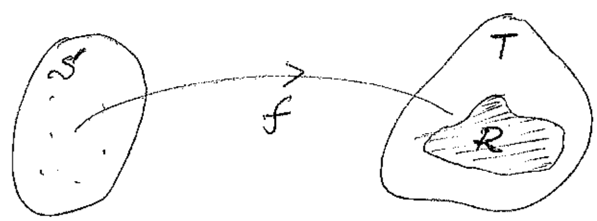
(d) $g: \mathbb{N} \to \mathbb{N}$ where g is defined by:

$$g(x) = \begin{cases} x+3 & x \geq 5 \\ x & x \leq 5 \end{cases}$$ No 5?

3-3-2 - Special functions:

* **Onto Functions; Surjective:**

Let $f: S \to T$ - we define the range of f, denoted R as a ~~the~~ subset of T, and R is the set of all images of elements of S under f.



If the range coincides with the codomain then the fct is called onto or surjective.

Examples: in previous example:

(a) (b) is surjective.

(b) Cities in the ~~coun~~ world and countries - ...

* **One-to-One functions; Injective:**

A fct $f: S \to T$ is 1-to-1 or injective, if no member of T is the image under f ~~of~~ two distinct elements of S

Examples:

$g: \mathbb{R} \to \mathbb{R}$ $g(x) = x^3$ 1-1

$h: \mathbb{R} \to \mathbb{R}$ $h(x) = x^2$ not injective.

$k: \mathbb{N} \to \mathbb{N}$ $k(x) = x^2$ is injective.

* **Bijective functions:** These fcts are both surj and injective.

* Invertible functions:

Let $f : S \to T$ be a function (mapping from S to T).
If the inverse mapping from T to S defines a function we say that f is invertible.

Theorem 3-2: A fct f is invertible if and only if f is a bijection.

expl: (i) $f : \mathbb{N} \to \mathbb{N}$

$$f(x) = x \Rightarrow f^{-1}(y) = y.$$

(ii) $f : \mathbb{R} \to \mathbb{R}$. $f(x) = a.x \quad (a \in \mathbb{R}, a \neq 0)$

$$f^{-1}(y) = \frac{y}{a}$$

HW #3:

page 120 Pbs: #6, #7, #8, #9, #10,
" 121 " : #12
" 122 " : #19, #20, #23, #24, #25