COMBINATORICS

Combinatorics is the branch of Dis. Math. that deals with counting. It Seeks to answer the question How many ways are there to .... Is it possible to .... How many operations --- How many words can we .... How many members are there in a set that.... Sometimes this trivial question is hard to answer.

4.1 Fundamental Principle of counting:- Multiplication Principle.

Definition: If there are n_1 ways for a first event and then n_2 ways for the second event, there are $n_1 \cdot n_2$ possible ways for the Sequence of the two events.

By Induction, the multiplication principle can be extended to a Sequence of any finite # of events.

Examples: ① The last part of your telephone number contains 4-digits How many 4-digit numbers are there?

$10 \cdot 10 \cdot 10 \cdot 10 = 10000$.

② Platenumber 3 letters and 3 digits? $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10$.

③ If a man has 4 suits, 8 shirts, and 5 ties how many outfits can he put together? $4 \cdot 8 \cdot 5 = 160$

④ How many ways are there to choose 3 officers from a club of 25 people? $25 \cdot 25 \cdot 25 = 15,625$

⑤ Consider No repetition in
① → $10 \cdot 9 \cdot 8 \cdot 7 = 5040$
② → $26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8$
④ → $25 \cdot 24 \cdot 23 = 13,800$

4.2 Selecting Elements from a Set:

Now we are interested in counting only a selection from set S. We can allow the selection to contain repetition or we can insist that it does not contains repetitions. We can also be interested in the order of the selection. If given $S = \{x, y, z, w\}$ and should select 3 elements - with repetition allowed and order counted. The selection will include xyx and xzz and yyy and zzx.

Finally, All together we have four selection methods.

	order matters	order does not matter
repetition allowed	K-Sample	K-Selection
repetition not allowed	K-Permutation	K-combination

Definitions:

① K-samples: Select k-things from a set S. Allow repetitions, but distinguish among lists in which the same elements appear in different order.

The list of k-sample selection contains n^k if $|S| = n$, and k the size of elements or thing selected.

exple: ① $S = \{a, b, c\}$ $k = 2$

the k samples are:

aa, ab, ba, bb
bc, cb, cc, ac
ca.

② $S = \{x, y, z, w\}$ $k = 3$.

xxx ....

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② K-selections: Select k-things from a set S. Allow repetitions, but equate lists containing the same elements in any order.

Exple $S = \{a, b, c\}$ $k = 2$

the K-selections are

aa ab
bb bc
cc ac

The list contains ? later.

③ K-Permutation: Select k things from a set S. Do not allow repetitions, and distinguish list by the order in which elements are selected.

Exple: ab bc ac
ba cb ca

④ K- Combinations: Select k-things from a set S. do not allow repetitions, equate lists containing the same elements in any order.

exple: ① $S = \{a, b, c\}$ $k=2$ ab, ac, bc

② $S = \{x, y, z, w\}$, $k=3$ xyz, xyw, yzw, xzw

Counting Formulas:

① K-Samples: Apply the principle of multiplication.

k positions choose anytime any of the n elements.

$$\underbrace{n \cdot n \cdot n \cdot \ldots \cdot n}_{k \text{ times}} = n^k.$$

② k-Permutations: we use the notation $P(n,k)$

Let $|S| = n$, then the Permutation of n things taken k at a time is $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)$

$$P(n,k) = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

where: $n! = n \cdot (n-1)(n-2) \ldots 3 \cdot 2 \cdot 1$

③ k-Combinations: denoted $C(n,k)$ and reads "combination of n things taken k at a time".

To count it we notice that to each k-combination there are many k-permutations. For each k-combinations there are $k! = k \cdot (k-1)(k-2) \ldots 2 \cdot 1$ hence $P(n,k) = C(n,k) \cdot k!$

$$\Rightarrow \quad C(n,k) = \frac{n!}{(n-k)!\,k!}$$

④ k-Selections: denoted by $S'(n,k)$

Theorem 4.1: $\forall n \geq 1$, $\forall k \geq 1$ the number of k-selections from a set with n elements is given by: $S'(n,k) = C(n+k-1, k)$

Applications:

1 How many 3-letter words (not necessarily meaningful) can be formed from the word "COMPILER" if letters:

(i) can be repeated $8 \cdot 8 \cdot 8$

(ii) do not repeat $\frac{8!}{5!}$ or $8 \cdot 7 \cdot 6 = 336$.

~~2 In How many ways can a president and a Vice President be selected from a group of 20 people.~~

2 In how many ways can 6 people be seated in a row of 6 chairs?

$6! = 720$.

3 ~~In how many ways can~~ A committee of 8 students is to be selected from a class consisting of 19 freshmen, 34 sophomores.

(a) In how many ways can 3 freshmen and 5 sophomores be selected?

(b) " " " " " a committee of exactly one freshman be selected?

(c) " " " " " " at most " " " selected.

(d) " " " " " " with at least 1 freshman be ".

a) multiplication Principle. $\Rightarrow$ $C(19,3)$ to choose the 3 freshman.
$C(34,5)$ " " " 5 sophomores.

$$\Rightarrow C(19,3) \cdot C(34,5) = \frac{19!}{3!\,16!} \cdot \frac{34!}{5!\,29!} = (969)(278{,}256).$$

b) $C(19,1) \cdot C(34,7) = 19\,(5{,}379{,}616)$

c) $C(19,1) \cdot C(34,7) + C(34,8) = \cdots\cdots$

d) Either count $C(19,1) \cdot C(34,7) + C(19,2) \cdot C(34,6) + \cdots + C(34,8)$
or $C(53,8) - C(34,8)$

4 A congressional committee of 3 is to be chosen from a set of 5 Democrats, 3 Republicans, and 4 Independents.

(i) In How many ways can the committee be chosen? $C(12,3)$

(ii) " " " " " " " " " If it cannot include both Dem. & Rep.

$$\underset{\text{no democrats}}{C(7,3)} + \underset{\text{no Rep.}}{C(9,3)} - \underset{\text{all indep. so as not to count twice.}}{C(4,3)}$$

(iii) In how many ways can the Committee be chosen if it must include at least 1 indep.

3 ways $\binom{4}{1}\binom{8}{2} + \binom{4}{2}\binom{8}{1} + \binom{4}{3} = 164$

4-3. Patterns: Sometimes we may have to extract some ~~information~~ collection of objects that are not necessarily all distinct. Example, How many words can we form from the set: M, I, S, S, I, S, S, I, P, P, I ., or , V, O, l, V, O ... etc.

Theorem 4-2: suppose there are objects, of which n_1 objects are of type 1, and n_2 of type 2, and ..., n_r of type r. Then there are

$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_r!}$$

Exple. # of words in MISSISSIPPI is: $\frac{11!}{4!\,4!\,2!}$

Examples: 1, 2, 3, 4

Prob 6 page 160:

(a) $4 \times 3 \times 2 = 24$

(b) $\binom{4}{2} + \binom{4}{2} = 12$ or $\frac{1}{2}$ the numbers.

(d) #'s > 200 $\Rightarrow$ $24 - (\#\text{ less } 200) = 24 - \binom{3}{2} = 24 - 6 = 18$

(c) #s odd = 12, or $\frac{1}{2}$.

HW #4 Page 159 # 1
" 160 # 7
" 161 # 9, 13, 15, 18
" 162 # ~~22~~, 23, 24, 25