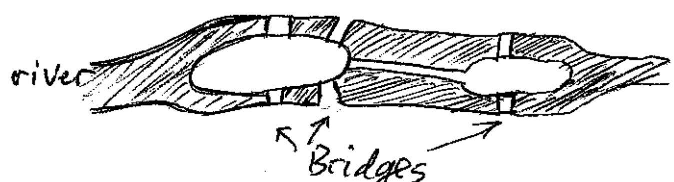
Lecture # V

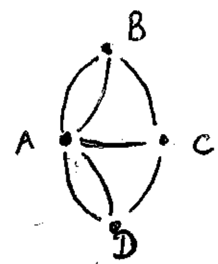
UNDIRECTED GRAPHS

A graph is a visual representation of certain data items and the connection between some of these items. A surprising number of real world situations can be represented as graphs — Organization chart, road maps, transportation and communication networks, ... — Historically, the theory originated by the 18th century mathcian L. Euler to solve the problem Known as the Königsberg bridge Problem:



7 bridges
2 islands.
and banks of the river.

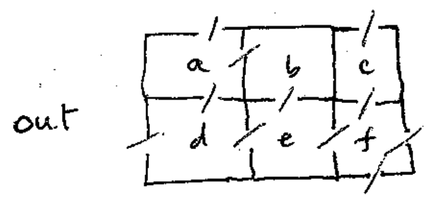
Find a path that crosses each bridge exactly once.
To model this problem as a multigraph

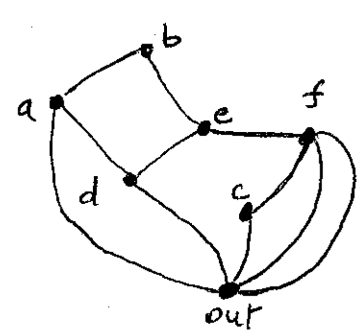


A, B, C, D are vertices or nodes
connection lines "bridges" are edges or arcs

Directed graph with arrows — Undirected graph without arrows —
Multigraph ⇒ multiedges between vertices.

Is it possible to walk in and out of each room in the house shown in the figure so that each door of the house is used exactly once?



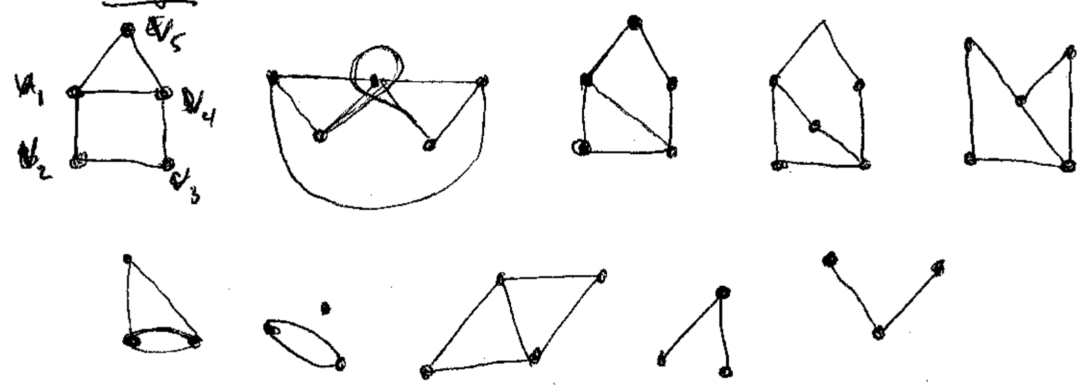


5.1 Simple Graphs: Graphs that have at most one edge between a pair of vertices. A simple graph is a pair $G = (V, E)$ where:

1. V is finite set whose elements are called vertices.
2. E is an irreflexive, symmetric relation on V.

the ordered pairs in E are called Edges. Irreflexivity implies no loops (edge from one vertex to itself). The symmetry implies adjacencies. Therefore $e = (u, v) = (v, u) = \{u, v\}$. ← u, v are endpoints of e.

Example:



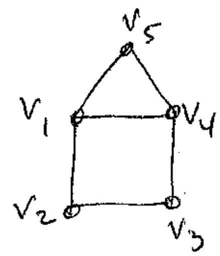
Symmetry and irreflexivity involve the matrix representation; if:

$V = (v_1, v_2, \dots v_n)$, M: $n \times n$ (logical) relation matrix for E

M: refered to as adjacency matrix. Diagonal entries are all F's and symmetric.

$$
\begin{array}{c|ccccc}
 & v_1 & v_2 & v_3 & v_4 & v_5 \\
\hline
v_1 & F & & & & \\
v_2 & T & F & & & \\
v_3 & F & T & F & & \\
v_4 & T & F & T & F & \\
v_5 & T & F & F & T & F
\end{array}
$$

for

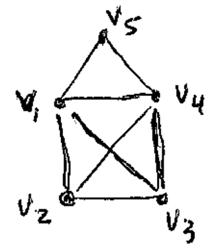


Let $G = (V, E)$ be a graph, let u, v be vertices. The <u>degree</u> of v denoted by $\delta(v)$ is the number of edges incident to v.

<u>**Theorem 5.1:**</u> Let $G = (V, E)$ be a graph.

$$\sum_{v \in V} \delta(v) = 2\,|E|$$

· Example:



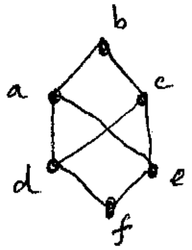
$E = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_2, v_4\}$
$\{v_3, v_4\}, \{v_4, v_5\}\}$.

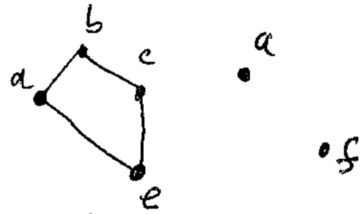
$|E| = 8 \Rightarrow \sum_{i=1}^{5} \delta(v_i) = \delta(v_1) + \delta(v_2) + \delta(v_3) + \delta(v_4) + \delta(v_5)$

$= 4 + 3 + 3 + 4 + 2$

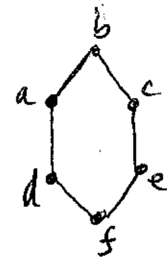
$2|E| = 2 \times 8 = \underline{\underline{16}} \qquad = \underline{\underline{16}}$

A subgraph of a graph $G = (V, E)$ is a graph $G' = (V', E')$ s.t $V' \subseteq V$ and $E' \subseteq E$. The subgraph G' is a spanning subgraph if $V = V'$.





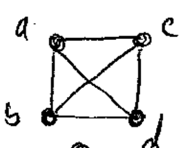
subgraph



Spanning subgraph.

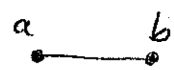
A graph is said to be complete if for all vertices $u, v \in V$, $\{u, v\} \in E$
A complete graph with n vertices is denoted by K_n.

Exple:



①

②



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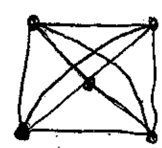


④

If $|V| = n$ in $G = (V, E)$ which is complete then the total number of degrees is $n(n-1) = n^2 - n$

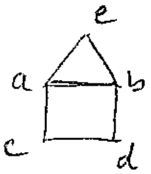
Exple: in ④ $n = 5 \Rightarrow$ Total # of degrees $= 5(5-1) = \underline{\underline{20}}$
K_5

Paths, Cycles, and Connectivity:

A path of length K in a graph is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that $\{v_{i-1}, v_i\} \in E$ for $i = 1, 2, \ldots, K$.

A path in an undirected graph is called a cycle (or circuit) if the first and the last vertexes in the path are the same vertex.

4/. Exple:

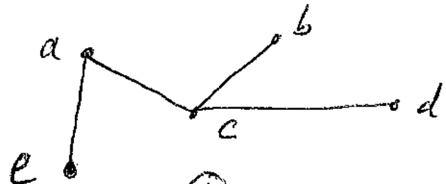


3 cycles a b c d a
a e b d c a
a e b a

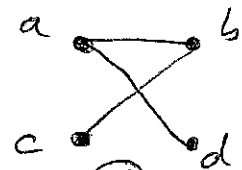
A cycle must have at least 3 edges.

a graph that does not contain any cycle is called acyclic.

Exple:



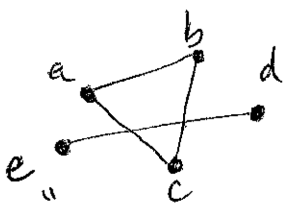
(i)



(ii)

A graph is said to be connected if there is a path between every pair of vertices in the graph.

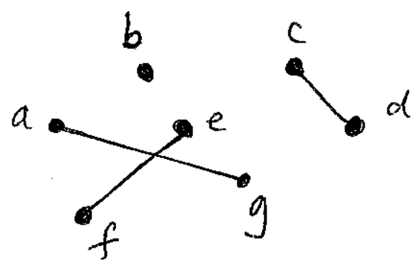
Exple: (i) & (ii) are connected.



not connected.

"The Phrase is connected by path" defines an equivalente relation denoted C. C is partitioned into equivalence classes subgraphs called <u>connectivity components</u> of the graph.
$C(G)$ denotes the connectivity number (number of classes).
If a graph is connected, $C(G) = 1$.

Exple:

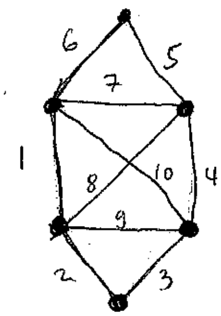


$C(G) = 4$

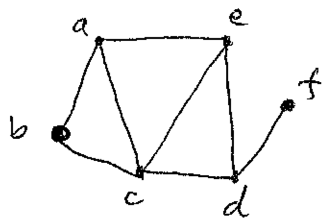
Eulerian Path: Theorem 5-2: A connected graph with at least two vertices has an Eulerian path iff there are 0 or 2 vertices of odd degree. The path is a cycle iff each vertex has even degree.

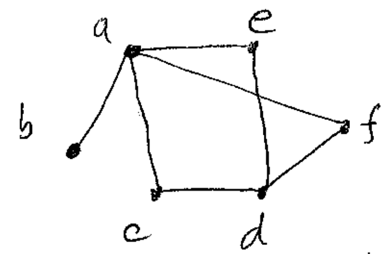
Examples:

1) Königsberg bridge problem.
2) In and Out house problem.
3)



Hamiltonian Circuits: It is a cycle that passes through each vertex exactly once

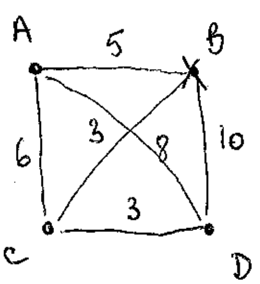


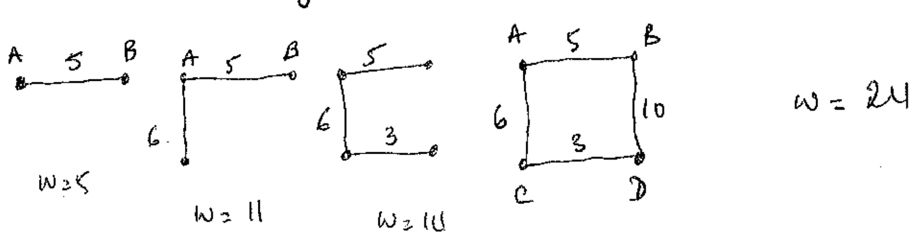


In A connected graph there is always a Hamiltonian circuit.

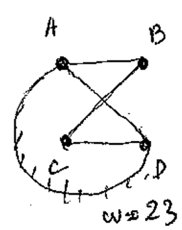
If the edges are assigned positive weights the problem is called Traveling Salesman problem. There are algorithms that find suboptimal solution. This means they guarantee finding ~~the best circuit~~ a Hamiltonian circuit that tends to be better than most

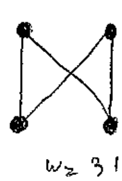
Example: The nearest neighbor algorithm

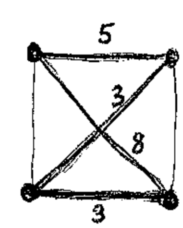




compare with

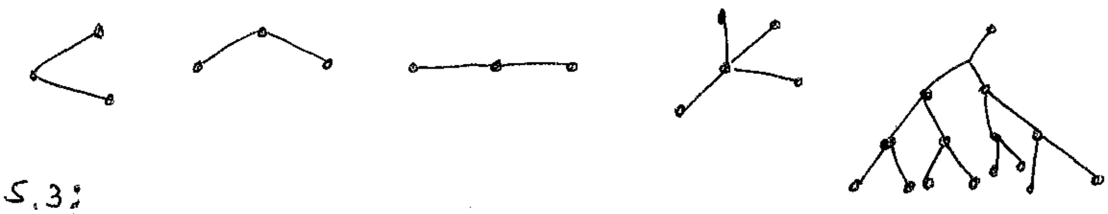






5.2 Trees: A graph $G = (V, E)$ is called a tree if G is connected and acyclic. They form an extremely important class of graphs:

Exples:



Theorem 5.3:

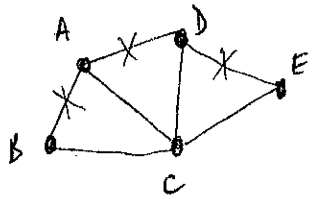
The following statements are equivalent for $G = (V, E)$, with n vertices and m edges.

1. G is a tree
2. There $\exists$ exactly one path between any pair of vertices in G
3. G is connected and $m = n - 1$
4. G is " and removing any edge disconnects G.
5. G is acyclic and $m = n - 1$
6. G is acyclic and adding any edge creates a cycle.

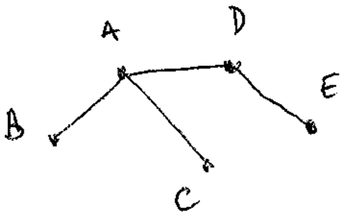
From this theorem we can assert that every connected graph contains a spanning tree

Two Procedures: 1) Deleting edges (that breaks the cycles mainly)

exple:



2) Opposite direction. Add edges so, that, no cycle has been created.



Minimal Spanning Trees: Highways between cities, and minimize the weight (# of edges). Algorithm for optimal solution is known.

Binary Rooted Trees: Decision Structure.

It is a tree with vertex set V such that one vertex $v \in V$ is the root of the tree, with unique path from v to u. The length of the path is called the level of the vertex u in the tree. Descendants of the root are those vertices that are adjacent to the root.

Examples: of Guessing problem.
and Binary Vectors.

Reading Assgnt.
§ 5-1 & 5-2.