

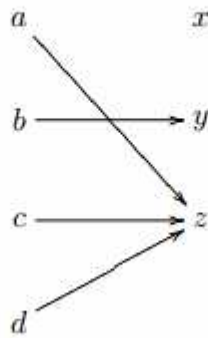
Question 1.

TASK: Let $A = \{ \dots \}$ and $B = \{ \dots \}$. The function f is defined by the relation pictured below. This function is [list of variations of options]

Variations:

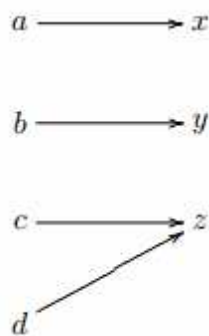
a)

EXAMPLE 2.2.1. Let $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$. The function f is defined by the relation pictured below. This function is neither injective nor surjective.



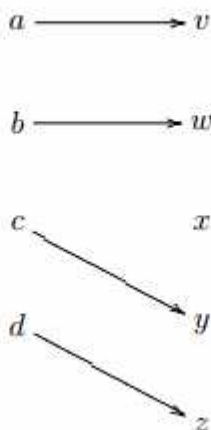
b)

EXAMPLE 2.2.2. $f: A \rightarrow B$ where $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$ defined by the relation below is a surjection, but not an injection.



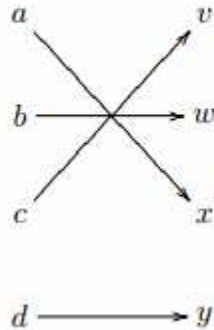
c)

EXAMPLE 2.2.3. $f: A \rightarrow B$ where $A = \{a, b, c, d\}$ and $B = \{v, w, x, y, z\}$ defined by the relation below is an injection, but not a surjection.



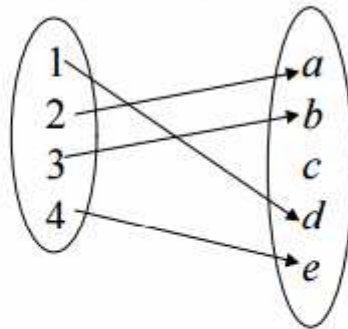
d)

EXAMPLE 2.2.4. $f: A \rightarrow B$ where $A = \{a, b, c, d\}$ and $B = \{v, w, x, y\}$ defined by the relation below both a surjection and an injection, and therefore a bijection. Notice that for a function to be a bijection, the domain and codomain must have the same cardinality.

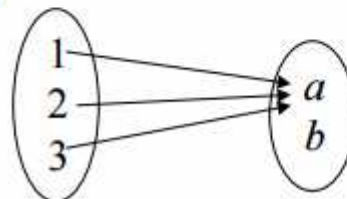


e) f)

- The function below is 1-1:

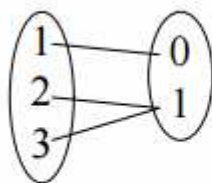


This function is not:

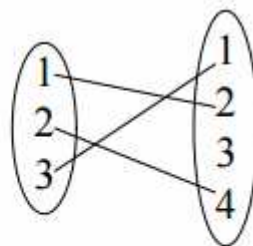


g)h)

- Examples:



ONTO



NOT ONTO

Question 2:

TASK : answer if the statement is true or false.

..... [answer box] (possible answer T,true, yes, Y, no, N)

variations of task:

examples :

1) surjective functions:

a) $f: \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = 2x + 1$

b) $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \quad f(x) = x^2$

c) $f: \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = 3x^3 - x$

2) NOT surjective functions:

a) $f: \mathbb{R} \rightarrow \mathbb{R} \quad h(x) = x^2$

b) $h: \mathbb{R} \rightarrow \mathbb{R} \quad h(x) = \sin x$

3) injective functions:

a) $f: \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = 2x + 1$

b) $f: \mathbb{R} \rightarrow \mathbb{R}$ for this domain all the linear functions are injective $f(x) = ax + b$

c) $h: \mathbb{R}^+ \rightarrow \mathbb{R} \quad h(x) = x^2$

4) NOT injective functions:

a) $h: \mathbb{R} \rightarrow \mathbb{R} \quad h(x) = x^2$

b) $h: \mathbb{R} \rightarrow \mathbb{R} \quad h(x) = \sin x$

5) bijective functions:

a) $h: \mathbb{R} \rightarrow \mathbb{R} \quad h(x) = x^3$

b) $h: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \quad h(x) = x^2$

c) $h: \mathbb{R} \rightarrow \mathbb{R} \quad h(x) = x$

d) $h: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \quad h(x) = \text{Sqrt}[x]$

Question 3. Proof:

Prove that:

variations :

a)

Proving Functions Are 1-1

- If $f: \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = 3x + 7$, prove it is one-to-one.
- Proof: Assume $f(a) = f(b)$. Show $a = b$.
Now $f(a) = f(b)$ means $3a + 7 = 3b + 7$, so $3a = 3b$, therefore $a = b$.

b)

Example 11

Let $A = \{x \in \mathbb{R} \mid x \neq 3\}$ and a function $f: A \rightarrow \mathbb{R}$ be defined by $f(x) = \frac{2x}{x-3}$. Decide and show whether the function f is an injective function.

Solution

~~What does the term injective mean?~~

~~One to one. If we show that the given function is one to one then we can conclude that it is an injective function. How do we show f is one to one?~~

~~By using definition (3.2). We assume $f(x) = f(y)$ and need to show that $x = y$.~~

~~What is $f(x)$ and $f(y)$ equal to in this case?~~

$$f(x) = \frac{2x}{x-3} \text{ and } f(y) = \frac{2y}{y-3}$$

Therefore $f(x) = \frac{2x}{x-3} = \frac{2y}{y-3} = f(y)$. We have

$$\frac{2x}{x-3} = \frac{2y}{y-3}$$

$$2x(y-3) = 2y(x-3)$$

$$2xy - 6x = 2xy - 6y$$

$$-6x = -6y$$

$$x = y$$

~~[Multiplying by $(y-3)(x-3)$]~~

~~[Expanding Brackets]~~

~~[Simplifying]~~

~~[Dividing through by -6]~~

Since we have $x = y$ therefore we conclude that the given function, f , is a one to one function or an injective function. ■

c)

Example 13

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^2$. Prove that the function f is **not** injective.

Solution

*What does the term **not injective** mean?*

We need to show that the given function, $f(x) = x^2$, is **not** one to one.

*How do we show that $f(x) = x^2$ is **not** one to one?*

By using definition (3.2). We assume $f(x) = f(y)$ and show that this does **not** lead to $x = y$. *What does $f(x) = f(y)$ mean in this case?*

It means that $x^2 = y^2$ because $f(x) = x^2$ and $f(y) = y^2$. Hence we have

$$x^2 = y^2$$

$$\sqrt{x^2} = \sqrt{y^2} \Rightarrow x = \pm y$$

Remember $x = \pm y$ means $x = y$ or $x = -y$. This time we **cannot** dismiss the answer $x = -y$ because the domain, \mathbb{R} , is the set of all real numbers and therefore both, $x = -y$ and $x = y$, are in the domain.

Since we do **not** have $x = y$ as the only solution therefore we conclude that the given function, f , is not one to one or it is **not** an injective function. ■

d)

Example 14

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $f(x) = x + 1$. Decide and show whether the function f is a surjective function.

Solution

Let y be in the codomain then we have $y = f(x) = x + 1$. We need to solve this equation, $y = x + 1$, for x . *What is x equal to?*

$$x = y - 1$$

The element $y \in \mathbb{R}$. *How do we know this?*

Because y is a member of the codomain and the codomain is equal to \mathbb{R} . Since y is real number then so is $y - 1$ which means x is a real number because $x = y - 1$.

Therefore x is in the domain. *Why?*

Because the domain is the set of all real numbers, \mathbb{R} .

Hence we have found an x in the domain such that $y = f(x)$ so the given function f is a surjective (or onto) function. ■

e)

Example 15

Let $A = \{x \mid x \in \mathbb{R} \text{ and } 0 \leq x \leq 5\}$ and $B = \{x \mid x \in \mathbb{R} \text{ and } 1 \leq x \leq 6\}$ and $f: A \rightarrow B$ be a function defined by $f(x) = x + 1$. Prove that the function f is a surjective function.

Solution

Let y be in the codomain. We need to find an x in A such that $y = f(x) = x + 1$ where y is in B . *How do we know y is in B ?*

Because B is the codomain. *What is y ?*

Since $y \in B = \{x \mid x \in \mathbb{R} \text{ and } 1 \leq x \leq 6\}$ therefore y is a real number between 1 and 6 (inclusive). *Can we find an x in A for such a y ?*

$$y = x + 1$$

$$x = y - 1$$

Since $1 \leq y \leq 6$ therefore $x = y - 1$ yields $0 \leq x \leq 5$. We are given

$A = \{x \mid x \in \mathbb{R} \text{ and } 0 \leq x \leq 5\}$. *Is x in the domain A ?*

Yes because $A = \{x \mid x \in \mathbb{R} \text{ and } 0 \leq x \leq 5\}$ which means A is the set of all real numbers between 0 to 5 (inclusive).

Therefore we have an x in A such that $x = y - 1$ which means that we have located an x in the domain with the property $y = f(x) = x + 1$.

We conclude that the given function is a surjection on the stated domain and codomain. ■

f)

Let $A = \{x \in \mathbb{R} \mid x \neq 3\}$ and $B = \{x \mid x \in \mathbb{R} \text{ and } x \neq 2\}$ and $f : A \rightarrow B$ be a function defined by $f(x) = \frac{2x}{x-3}$. Prove that the function f is an onto function.

Solution

Remember an onto function, surjective function and surjection are all the same thing. To prove that the given function is onto we show that there is an x in A such that $f(x) = y$

$$y = f(x) = \frac{2x}{x-3}$$

Need to solve this equation, $y = \frac{2x}{x-3}$, for x :

$$y(x-3) = 2x \quad \left[\text{Multiplying both sides by } (x-3) \right]$$

$$yx - 3y = 2x \quad \left[\text{Expanding Brackets} \right]$$

$$yx - 2x = 3y \quad \left[\text{Collecting Like Terms} \right]$$

$$x(y-2) = 3y \quad \left[\text{Factorizing} \right]$$

$$x = \frac{3y}{y-2} \quad \left[\text{Dividing through by } (y-2) \right]$$

$x = \frac{3y}{y-2}$ is a real number provided $y \neq 2$. Hence we have located an x in

$A = \{x \in \mathbb{R} \mid x \neq 3\}$ such that $f(x) = y$. We conclude that f is an onto (or surjective) function. ■

g)

(i) Suppose we wanted to prove injectivity of the function $f : \mathbb{N} \rightarrow \mathbb{N}$ defined by the rule $\forall n \in \mathbb{N} \ f(n) := n + 3$. How to proceed? Very simple. Just use (7.1) directly:

Let $a, b \in \mathbb{N}$ and assume $f(a) = f(b)$. Thus you are assuming $a + 3 = b + 3$. It follows immediately that $a = b$. Therefore f is injective.

h)

EXAMPLE 2.4.1. Prove that the function $g : \mathbb{N} \rightarrow \mathbb{N}$, defined by $g(n) = \lfloor n/3 \rfloor$, is surjective.

PROOF. Let $n \in \mathbb{N}$. Notice that $g(3n) = \lfloor (3n)/3 \rfloor = \lfloor (3n)/3 \rfloor = n$. Since $3n \in \mathbb{N}$, this shows n is in the range of g . Hence g is surjective. □

Question 4:

TASK: let $f(x) = \dots$ and $g(x) = \dots$ find the compositions of functions $g \circ f$ and $f \circ g$

..... = [variations of answers]

examples:

a)

Example 138 Let $g(x) = e^{x+1}$ and $f(x) = x - 3$ be functions defined on the real line. It follows that $g(f(x)) = e^{f(x)+1} = e^{x-2}$ and $f(g(x)) = g(x) - 3 = e^{x+1} - 3$.

b)

Example 2:

- Let f and g be two functions from Z to Z , where
 - $f(x) = 2x$ and $g(x) = x^2$.
 - $f \circ g : Z \rightarrow Z$
 - $(f \circ g)(x) = f(g(x))$
= $f(x^2)$
= $2(x^2)$
 - $g \circ f : Z \rightarrow Z$
 - $(g \circ f)(x) = g(f(x))$
= $g(2x)$
= $(2x)^2$
= $4x^2$
- Note that the order of the function composition matters

c)

- Given $f(x) = 2x + 3$ and $g(x) = -x^2 + 5$, find $(f \circ g)(x)$.

In this case, I am not trying to find a certain numerical value. Instead, I am trying to find the formula that results from plugging the formula for $g(x)$ into the formula for $f(x)$. I will write the formulas at each step, using parentheses to indicate where the inputs should go:

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(-x^2 + 5) \\ &= 2(\quad) + 3 \quad \dots \text{setting up to insert the input formula} \\ &= 2(-x^2 + 5) + 3 \\ &= -2x^2 + 10 + 3 \\ &= -2x^2 + 13\end{aligned}$$

d)

- Given $f(x) = 2x + 3$ and $g(x) = -x^2 + 5$, find $(g \circ f)(x)$.

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2x + 3) \\ &= -(\quad)^2 + 5 \quad \dots \text{setting up to insert the input} \\ &= -(2x + 3)^2 + 5 \\ &= -(4x^2 + 12x + 9) + 5 \\ &= -4x^2 - 12x - 9 + 5 \\ &= -4x^2 - 12x - 4\end{aligned}$$