Methods of Proof Examples

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We start with an example of a **direct proof**.

Proposition 1. If n is an odd integer, then n^2 is an odd integer.

Proof. Let n be an odd integer, then $n = 2k + 1$ for some integer k. Then,

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Since $n^2 = 2m + 1$ for the integer $m = 2k^2 + 2k$, we conclude that n^2 is odd. □

To prove a proposition in the form $p \to q$, it is sufficient to prove its logically equivalent contrapositive $\neg q \to \neg p$. This is called a **proof by contrapositive**.

Proposition 2. If n^2 is an even integer, then n is an even integer.

Proof. The contrapositive of

$$(n^2 \text{ is an even integer}) \to (n \text{ is an even integer})$$

is

$$\neg(n \text{ is an even integer}) \to \neg(n^2 \text{ is an even integer})$$

which is equivalent to

$$(n \text{ is an odd integer}) \to (n^2 \text{ is an odd integer})$$

which was proved in Proposition 1. □

A **proof by contradiction** works as follows. To prove p, start by assuming that p is false and deduce consequences. If you deduce a contradiction with something known or assumed to be true, then the initial assumption that p is false was wrong. Therefore, p must be true.

Proposition 3. $\sqrt{2}$ is an irrational number.

Proof. Assume that $\sqrt{2}$ is a rational number. Then, $\sqrt{2} = a/b$ for two positive integers a and b. Assume that a and b have no common factors so that the fraction a/b is an irreducible fraction. By squaring both sides of $\sqrt{2} = a/b$, we deduce $2 = a^2/b^2$. Therefore

$$2b^2 = a^2 \tag{1}$$

which implies that a^2 is even. From Proposition 2, we conclude that a is even, i.e., $a = 2k$ for some integer k. Substitute $a = 2k$ in equation (1) to get

$$b^2 = 2k^2.$$

We conclude that b^2 is even which implies that b is even. We have derived that both a and b are even but this a contradiction since we assumed that the fraction a/b was irreducible. Therefore, $\sqrt{2}$ is an irrational number. □

Let's now look an example of a **proof by cases**.

Proposition 4. There exists irrational numbers a and b such that a^b is rational.

Proof. Consider the number $\sqrt{2}^{\sqrt{2}}$ which is either rational or irrational.

Case 1. If $\sqrt{2}^{\sqrt{2}}$ is rational, by choosing $a = b = \sqrt{2}$ we get that a^b is rational.

Case 2. If $\sqrt{2}^{\sqrt{2}}$ is irrational, by choosing

$$a = \sqrt{2}^{\sqrt{2}} \quad \text{and} \quad b = \sqrt{2}$$

we get that a^b is rational since

$$a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2.$$

□

Observe that this proof does not tell us whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational. We used the fact that it is either rational or irrational.