W15 QMS 202 Lecture 4

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t Test of Hypothesis for the Mean (σ unknown)

- In almost all hypothesis-testing problems concerning the mean μ we do not know the population standard deviation σ.
- If the population standard deviation σ is unknown, you instead use the sample standard deviation S.
- Because of this change, you use the t distribution instead of the Z distribution to test the null hypothesis about the mean.
- When using the t distribution you must assume the population you are sampling from follows a normal distribution.
- All other steps, concepts, and conclusions are the same.

t Test of Hypothesis for the Mean (σ unknown)

- If we assume that the population is normally distributed, then the sampling distribution of the mean follows a t distribution with $n-1$ degrees of freedom and we use the t test for the mean.
- If the population is not normally distributed, we can still use the t test if the sample size is large enough for the Central Limit Theorem to take effect.
- To test the difference between the sample mean $\overline{X}$, and the population mean μ, when using the sample standard deviation S, we define the test statistic t_{STAT} by

$$t_{\text{STAT}} = \frac{\overline{X} - \mu}{\frac{S}{\sqrt{n}}}$$

- t_{STAT} test statistic follows a t distribution with $n-1$ degrees of freedom.

Critical Value Approach - Example 1

- The average cost of a hotel room in New York is said to be \$168 per night. To determine if this is true, a random sample of 25 hotels is taken and resulted in an $\overline{X}$ of \$172.50 and an S of \$15.40. Test the appropriate hypotheses at $\alpha = 0.05$. Assume that the population distribution is normal.

1. State the appropriate null and alternative hypotheses.
   - $$H_0 : \mu = 168, \qquad H_1 : \mu \neq 168$$
2. Specify the desired level of significance and the sample size.
   - We are given $\alpha = 0.05$ and $n = 25$. So, $df = 25 - 1 = 24$.
3. Determine the appropriate technique
   - σ is unknown so this is a t test.

Critical Value Approach - Example 1

4. Determine the critical values

- The critical values $\pm t_{\frac{\alpha}{2}}$ of the t distribution with $\alpha = 0.05$ and $df = 24$ can be found using the calculator.

i. Select **STAT**, **F5** (DISTR), **F2**(t) and then **F3**(Invt). Now in the **Inverse Student-t** menu select the following options:

ii. Data: **F2** (Variable)

iii. Area: **0.025 EXE**

iv. df: **24 EXE**

v. Press key **EXE**

- You will get xInv = 2.06389856. The calculator only gives you the upper tail area. The value for the lower tail area is -2.06389856 because the t-distribution is symmetric.

Critical Value Approach - Example 1

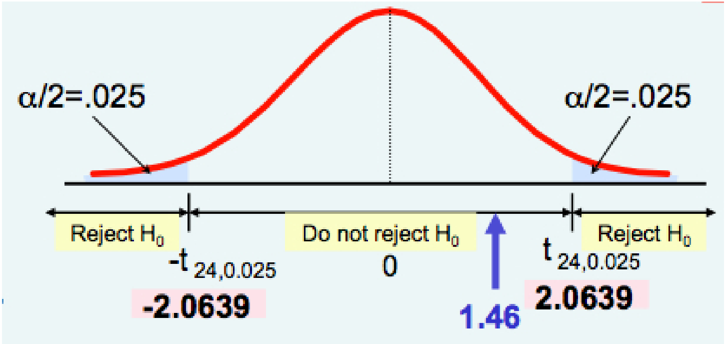
5. Collect the data and compute the test statistic

- Using the sample results: $n = 25$, $\overline{X} = 172.50$, $S = 15.40$
- We get the test statistic

$$t_{\text{STAT}} = \frac{\overline{X} - \mu}{\frac{S}{\sqrt{n}}} = \frac{172.50 - 168}{\frac{15.40}{\sqrt{25}}} = \frac{4.5}{3.08} = 1.46$$

Critical Value Approach - Example 1

6. Is the test statistic in the rejection region?



6. (continued) Reach a decision and interpret the result
   - Since $-2.0639 < t_{\text{STAT}} < 2.0639$, we do not reject the null hypothesis and conclude that there is insufficient evidence that the true mean cost is different from \$168.

p-Value Approach - Example 1 - Using CASIO Calculator

1. Select **STAT**, **F3** (TEST), **F2**(t) and then **F1**(1-S). Now select in the **1-Sample tTest** the following options:
2. Data: **F2** (Variable)
3. μ: **F1 EXE** ($\neq \mu 0$)
4. $\mu 0$: **168 EXE**
5. $\overline{x}$: **172.50 EXE**
6. sx: **15.40 EXE**
7. n: **25 EXE**
8. Save Res: If you do not want to save results in a list press **F1** (None). If you want to save results in a list press **F2** and type the number of list where you want to save results.
9. Execute: key **EXE** or **F1**(Calc).

- You will get the following summary:

p-Value Approach - Example 1- Using CASIO Calculator

- **1-Sample tTest**
- $\mu \neq 168$
- $t = 1.46103896$
- $p = 0.15697437$
- $\overline{x} = 172.5$
- $sx = 15.4$
- $n = 25$
- **Statistical Decision**: Since $p -$ value $= 0.157 > \alpha = 0.05$, we do not reject the null hypothesis.
- **Conclusion**: There is insufficient evidence that the true mean cost is different from $168.

p-Value Approach - Example 1- Further Conclusions

- The difference between the sample mean and the believed mean is $172.50 - 168 = 4.50$.
- The p-value indicates that if the null hypothesis is true, the probability that a sample of 25 hotel room prices in NYC could have a sample mean that differs by 4.5 or more from 168 is 0.157 Remember that the

$$p\text{-value} = P(t < -1.46) + P(t > 1.46) = 0.1573$$

since $t_{\text{STAT}} = 1.46$.

- In other words, if the mean cost per room is truly \$168, then there is 15.7% chance of observing a sample mean below 163.50 or above 172.50. Where

$$163.50 = 168 - 4.50 \quad \text{and} \quad 172.50 = 168 + 4.50$$

p-Value Approach - Example 1- Further Conclusions

- It would be incorrect to say that there is a 15.7% chance that the null hypothesis is true!
- This is because the p-value is a conditional probability computed under the assumption that the null hypothesis is true. In general, it is proper to say:
- If the null hypothesis is true, then there is a $(p - \text{value}) \times 100\%$ chance of observing a test statistic at least as contradictory to the null hypothesis as the sample result.

Connection of Two-Tail Tests to Confidence Intervals

- For $\overline{X} = 172.50$, $S = 15.40$ and $n = 25$, the 95% confidence interval for μ is
- $$\overline{X} \pm t_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} = 172.50 \pm 2.0639 \frac{15.40}{\sqrt{25}} = 172.50 \pm 6.3568$$
- or in the interval form $166.14 \leq \mu \leq 178.86$.
- Since this interval contains the hypothesized mean 168, we do not reject the null hypothesis at $\alpha = 0.05$.

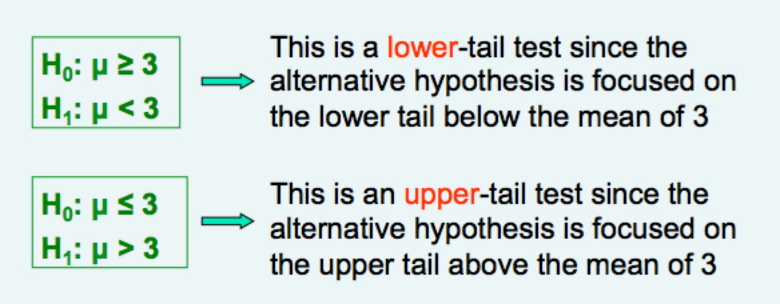
One-Tail Tests

- So far we studied two-tail tests where the null hypothesis was given in terms of $=$ sign

$$H_0 : \mu = 3, \qquad H_1 : \mu \neq 3$$

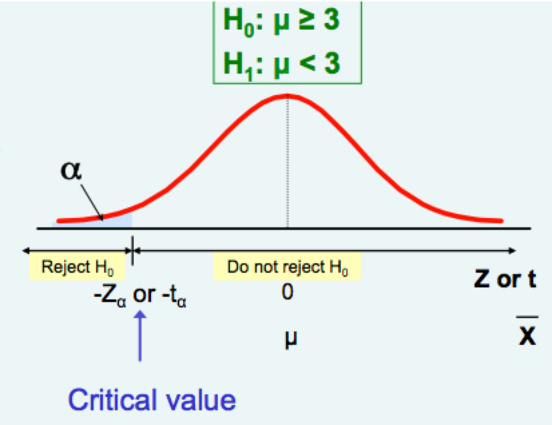
- The alternative hypothesis $H_1 : \mu \neq 3$ contains two possibilities:
  - either the mean is less than 3,
  - or the mean is greater than 3.
- For this reason, the rejection is divided into the two tails of the sampling distribution of the mean.
- In many cases, the alternative hypothesis focuses on a particular direction only. For example:
  - the population mean is less than a specified value,
  - the population mean is greater than a specified value.
- In this case we use **one-tail test**.

One-Tail Tests

- 

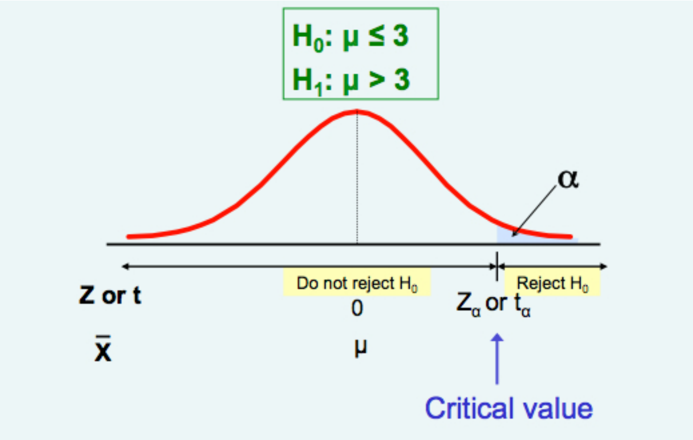
- The alternative hypothesis contains the statement you are trying to prove.

Lower-Tail Tests

- There is only one critical value, since the rejection area is in only one tail
- 

Upper-Tail Tests

- There is only one critical value, since the rejection area is in only one tail



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Upper-Tail Test-Critical Value Approach - Example 1

- A phone industry manager thinks that customer monthly cell phone bills have increased, and now average over \$52 per month. The company wishes to test this claim. (Assume a normal population)

1. State the appropriate null and alternative hypothesis.
   - $$H_0 : \mu \leq 52, \qquad H_1 : \mu > 52$$
   - $H_0 : \mu \leq 52$ means the average is not over \$52 per month
   - $H_1 : \mu > 52$ means the average is greater than \$52 per month (i.e., sufficient evidence exists to support the managers claim).
2. Specify the desired level of significance and the sample size.
   - Suppose that $\alpha = 0.10$ is chosen for this test and $n = 25$. So, $df = 25 - 1 = 24$.
3. Determine the appropriate technique
   - σ is unknown so this is a t test.

Upper-Tail Test-Critical Value Approach - Example 1

4. Determine the critical value
- The critical value t_α of the t distribution with $\alpha = 0.10$ and $df = 24$ can be computed using the calculator.

i. Select **STAT**, **F5** (DISTR), **F2**(t) and then **F3**(Invt). Now in the **Inverse Student-t** menu select the following options:

ii. Data: **F2** (Variable)

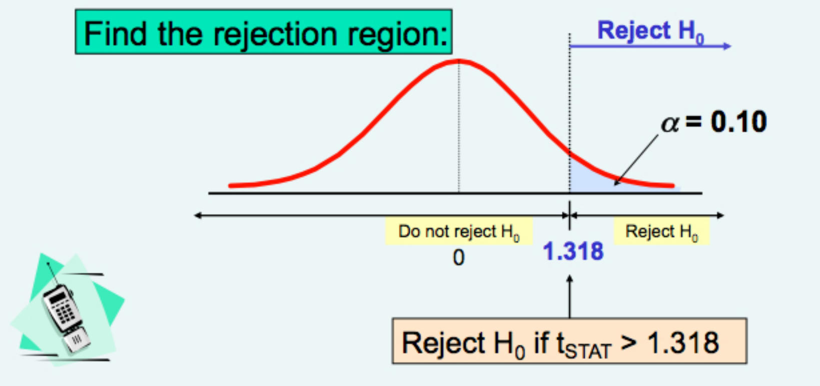
iii. Area: **0.10 EXE**

iv. df: **24 EXE**

v. Press key **EXE**

- You will get xInv = 1.31783593. So, the critical value $t_\alpha = 1.318$.
- **Remark**: If we had a Lower-Tail test, i.e., the alternative hypothesis was $H_1 : \mu < 52$, the critical value would be **negative**. In this case it would be $t_\alpha = -1.318$.

Upper-Tail Test-Critical Value Approach - Example 1

- There is only one critical value, since the rejection area is in only one tail
- 

Upper-Tail Test-Critical Value Approach - Example 1

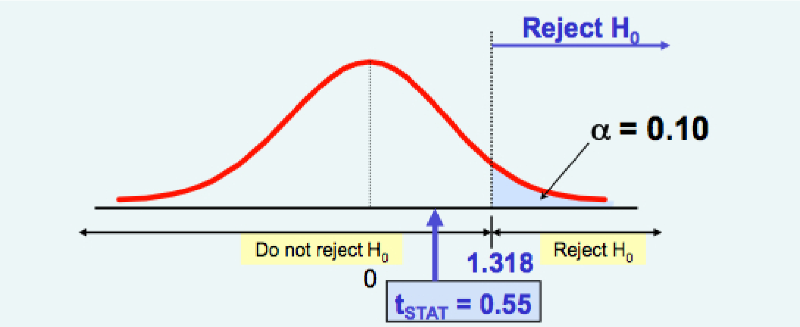
5. Collect the data and compute the test statistic

- Suppose a sample is taken with the following results: $n = 25$, $\overline{X} = 53.1$, $S = 10$
- Then the test statistic is

$$t_{\text{STAT}} = \frac{\overline{X} - \mu}{\frac{S}{\sqrt{n}}} = \frac{53.1 - 52}{\frac{10}{\sqrt{25}}} = \frac{1.1}{2} = 0.55$$

Upper-Tail Test-Critical Value Approach - Example 1

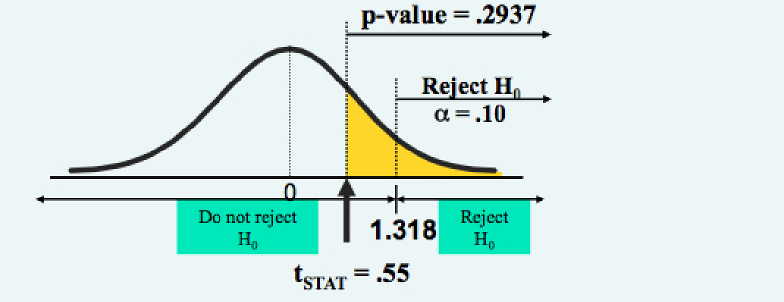
6. Is the test statistic in the rejection region?



6. (continued) Reach a decision and interpret the result
   - Since $t_{\text{STAT}} = 0.55 < 1.318$, we do not reject the null hypothesis and conclude that there is insufficient evidence that the true mean bill is over $52.

Upper-Tail Test *p*-Value Approach - Example 1

- Calculate the p-value and compare to α



- We will compute the *p*-value using the calculator. Remember that the

$$p\text{-value} = P(t > 0.55)$$

since $t_{\text{STAT}} = 0.55$. The calculator will give us *p*-value $= 0.2937$.

Upper-Tail Test p-Value Approach - Example 1 Using CASIO Calculator

1. Select **STAT**, **F3** (TEST), **F2**(t) and then **F1**(1-S). Now select in the **1-Sample tTest** the following options:
2. Data: **F2** (Variable)
3. μ: **F3 EXE** ($> \mu 0$)
4. $\mu 0$: **52 EXE**
5. $\overline{x}$: **53.1 EXE**
6. sx: **10 EXE**
7. n: **25 EXE**
8. Save Res: If you do not want to save results in a list press **F1** (None). If you want to save results in a list press **F2** and type the number of list where you want to save results.
9. Execute: key **EXE** or **F1**(Calc).

- You will get the following summary:

Upper-Tail Test p-Value Approach - Example 1 Using CASIO Calculator

- **1-Sample tTest**
- $\mu > 52$
- $t = 0.55$
- $p = 0.29370099$
- $\overline{x} = 53.1$
- $sx = 10$
- $n = 25$
- **Statistical Decision**: Since $p-\text{value} = 0.2937 > \alpha = 0.10$, we do not reject the null hypothesis.
- **Conclusion**: There is insufficient evidence that the true mean bill is over $52.

Lower-Tail Test p-Value Approach - Example 1

- A researcher is asked to test the hypothesis that the average price of a 2-star (CAA rating) motel room has decreased since last year. Last year study showed that the prices were normally distributed with an average of \$89.50. A random sample of twelve 2-star motels has yielded the following information on room prices in \$:
85.00	92.50	87.50	89.90
90.00	82.50	87.50	90.00
85.00	89.00	91.50	87.50
- If it is believed that the distribution of room prices is normal. What conclusion should the researcher make at the 5% level of significance?

Lower-Tail Test *p*-Value Approach - Example 1 Using CASIO Calculator

- First enter the data into **List 1**

1. Select **STAT**, **F3** (TEST), **F2**(t) and then **F1**(1-S). Now select in the **1-Sample tTest** the following options:
2. Data: **F1** (List)
3. μ: **F2 EXE** ($< \mu 0$)
4. $\mu 0$: **89.50 EXE**
5. List: **List 1 EXE**
6. Freq: **1 EXE**
7. Save Res: If you do not want to save results in a list press **F1** (None). If you want to save results in a list press **F2** and type the number of list where you want to save results.
8. Execute: key **EXE** or **F1**(Calc).

- You will get the following summary:

Lower-Tail Test p-Value Approach - Example 1 Using CASIO Calculator

- **1-Sample tTest**
- $\mu < 89.50$
- $t = -1.5915085$
- $p = 0.06990119$
- $\overline{x} = 88.1583333$
- $sx = 2.92029212$
- $n = 12$
- **Statistical Decision**: Since p-value $= 0.069 > \alpha = 0.05$, we do not reject the null hypothesis.
- **Conclusion**: The evidence does not indicate that the mean price of a 2-star motel room has decreased this year.

Hypothesis Tests for Proportion

- Hypothesis tests for proportion involve categorical variables.
- Two possible outcomes
  - Possesses characteristic of interest
  - Does not possess characteristic of interest
- Fraction or proportion of the population in the category of interest is denoted by π.

Hypothesis Tests for Proportion

- Sample proportion in the category of interest is denoted by p

$$p = \frac{X}{n} = \frac{\#\text{ of items having the characteristic of interest}}{\text{Sample Size}}$$

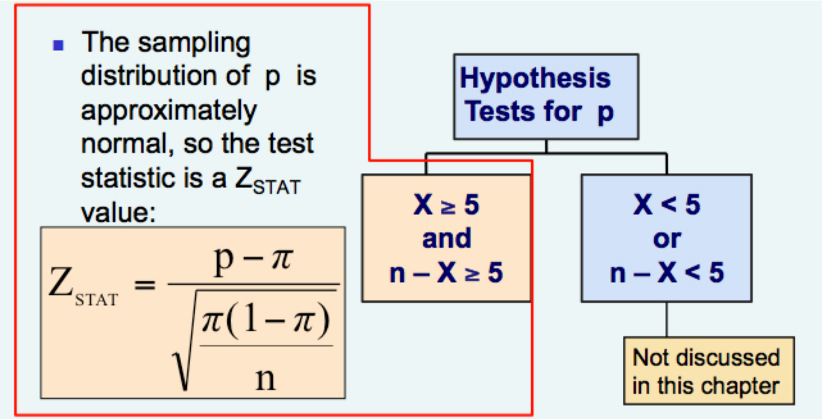
- When both X and $n - X$ are at least 5, p can be approximated by a normal distribution with
  - the mean and

$$\mu_p = \pi$$

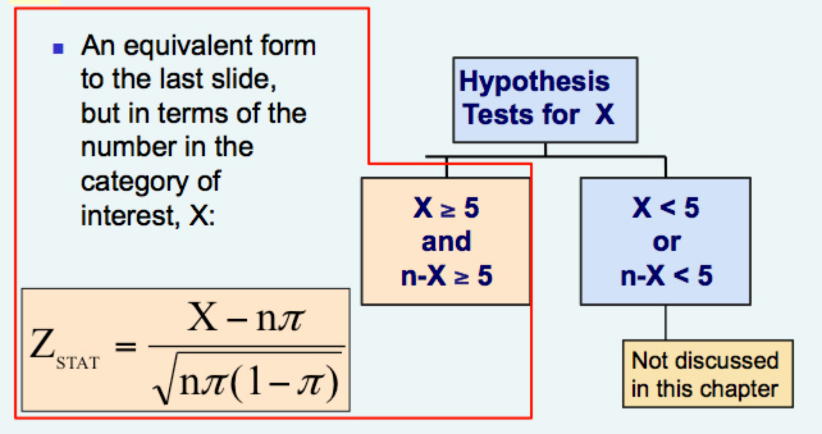
  - the standard deviation

$$\sigma_p = \sqrt{\frac{\pi(1-\pi)}{n}}$$

Z Test for Proportion in Terms of π



Z Test for Proportion in Terms of the Number of Events of Interest



Z Test for Proportion-Critical Value Approach-Example 1

- A marketing company claims that it receives responses from 8% of those surveyed. To test this claim, a random sample of 500 were surveyed with 25 responses. Test at the $\alpha = 0.05$ significance level.

1. State the appropriate null and alternative hypothesis.
   - $$H_0 : \pi = 0.08, \qquad H_1 : \pi \neq 0.08$$
2. Specify the desired level of significance and the sample size.
   - $\alpha = 0.05$ is chosen for this test and $n = 500$. Since $X = 25 > 5$ and $n - X = 475 > 5$, p can be approximated by a normal distribution.
   - Using $n = 500$ and $X = 25$ we get $p = \frac{X}{n} = \frac{25}{500} = 0.05$
3. Determine the appropriate technique
   - We use Z test.

Z Test for Proportion-Critical Value Approach-Example 1

4. Determine the critical values

- Using $\alpha = 0.05$ we get the critical values $Z_{\frac{\alpha}{2}} = \pm 1.96$

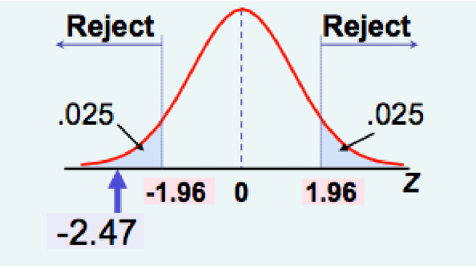
5. Compute the test statistic Z_{STAT}

- Using: $n = 500$, $p = 0.05$ and $\pi = 0.08$
- We get the test statistic

$$Z_{\text{STAT}} = \frac{p - \pi}{\sqrt{\frac{\pi(1-\pi)}{n}}} = \frac{0.05 - 0.08}{\sqrt{\frac{0.08(1-0.08)}{500}}} = \frac{-0.03}{0.01213} = -2.47$$

Z Test for Proportion-Critical Value Approach-Example 1

6. Is the test statistic in the rejection region?



6. (continued) Reach a decision and interpret the result
   - Since $Z_{\text{STAT}} = -2.473 < -1.96$, we reject the null hypothesis H_0 at $\alpha = 0.05$ and conclude that there is sufficient evidence to reject the company's claim of 8% response rate.

Z Test for Proportion-p Value Approach-Example 1 Using CASIO Calculator

1. Select **STAT**, **F3** (TEST), **F1**(Z) and then **F3**(1-p). Now select in the **1-Prop ZTest** the following options:
2. Prop: **F1** ($\neq p0$)
3. $p0$: **0.08 EXE**
4. x: **25 EXE**
5. n: **500 EXE**
6. Save Res: If you do not want to save results in a list press **F1** (None). If you want to save results in a list press **F2** and type the number of list where you want to save results.
7. Execute: key **EXE** or **F1**(Calc).

- You will get the following summary:

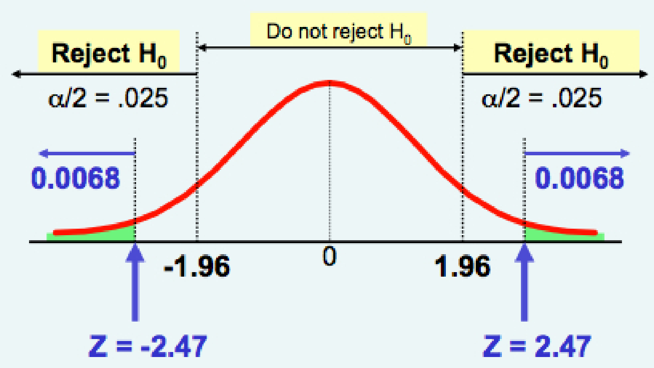
Z Test for Proportion- p Value Approach-Example 1 Using CASIO Calculator

- **1-Prop ZTest**
- Prop $\neq 0.08$
- $z = -2.4726768$
- $p = 0.01341053$
- $\overline{p} = 0.05$
- $n = 500$
- **Statistical Decision**: Since p-value $= 0.0134 < \alpha = 0.05$, we reject the null hypothesis.
- **Conclusion**: There is sufficient evidence to reject the company's claim of 8% response rate.

Z Test for Proportion- p Value Approach-Example 1 Using CASIO Calculator

- We have

  p-value $= P(Z < -2.47) + P(Z > 2.47) = 0.0068 + 0.0068 = 0.0136$

- 

Potential Pitfalls and Ethical Considerations

- Use randomly collected data to reduce selection biases.
- Do not use human subjects without informed consent.
- Choose the level of significance α, and the type of test (one-tail or two-tail) before data collection.
- Do not employ data snooping to choose between one-tail and two-tail test, or to determine the level of significance.
- Do not practice data cleansing to hide observations that do not support a stated hypothesis.
- Report all pertinent findings including both statistical significance and practical importance.